A QUASI-METRIZATION THEOREM FOR HYBRID TOPOLOGIES ON THE REAL LINE IN ZF

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Abstract. Hybrid topologies on the real line have been studied by various authors. Among the hybrid spaces, there are also the Hattori spaces. However, some of the hybrid spaces are not homeomorphic to Hattori spaces. In this article, a common generalization of at least four kinds of the hybrid topologies on the real line is described. In the absence of the Axiom of Choice, a quasi-metrization theorem for such hybrid spaces is proved. It is shown that Kofner’s quasi-metrization theorem for generalized ordered spaces is false in every model of $\mathbf{ZF}$ in which there exists an infinite Dedekind-finite subset of the real line.

1. Introduction

In this article, a 4-cover of the set $\mathbb{R}$ of real numbers is a collection $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$ of pairwise disjoint subsets of $\mathbb{R}$ such that $\mathbb{R} = A_1 \cup A_2 \cup A_3 \cup A_4$. Some members of a 4-cover may be empty.

Given a 4-cover $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$ of $\mathbb{R}$, for $x \in \mathbb{R}$, we define the family $\mathcal{B}(x)$ as follows:

$$\mathcal{B}(x) = \begin{cases} \{(x - \epsilon, x + \epsilon) : \epsilon > 0\} & \text{if } x \in A_1; \\ \{\{x\}\} & \text{if } x \in A_2; \\ \{[x, x + \epsilon) : \epsilon > 0\} & \text{if } x \in A_3; \\ \{(x - \epsilon, x] : \epsilon > 0\} & \text{if } x \in A_4. \end{cases}$$

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The unique topology $\tau_A$ on the real line $\mathbb{R}$ such that, for every $x \in \mathbb{R}$, the family $B(x)$ is a local base at $x$ in $\langle \mathbb{R}, \tau_A \rangle$ is called the hybrid topology determined by $A$. The topological space $H_4(A) = \langle \mathbb{R}, \tau_A \rangle$ will be called the hybrid space determined by $A$. Similar general constructions, leading to the concept of a generalized ordered space, appeared for instance, in [1], [21], [23] and [24]. Of course, $H_4(A)$ is a $T_3$-space and a generalized ordered space (see [1], [21], [23] and [24]). We notice that if $A_2 \cup A_4 = \emptyset$, the space $H_4(A)$ is the Hattori space $H(A_1)$; that is, in $H(A_1)$, a base of neighborhoods of $x \in A_1$ is the family of usual Euclidean open neighborhoods of $x$, and a base of neighborhoods of $x \in \mathbb{R} \setminus A_1$ is the family $\{[x, x+\epsilon) : \epsilon > 0\}$. If $A_1 = \mathbb{Q}$, $A_4 = \mathbb{R} \setminus \mathbb{Q}$ (in this case, $A_2 = A_3 = \emptyset$), our $H_4(A)$ is the Engelking-Lutzer line (see [8] and [21, p.122]).

We recall that Hattori spaces (called also $H$-spaces) were investigated, for instance, in [2]–[4] [13], [22], [26]. If $A_1 \cup A_2 = \emptyset$, the space $J(A_3)$ in [26] is our $H_4(A)$. The space $J(\mathbb{R} \setminus \mathbb{Q})$ appeared in [27, pp. 48, 166, 244, 281]. If $A_3 \cup A_4 = \emptyset$, the space $D_1(A_2)$ in [26] is our $H_4(A)$. If $A_1 \cup A_4 = \emptyset$, the space $D_2(A_2)$ in [26] is our $H_4(A)$. If $A_1 \cup A_2 \cup A_4 = \emptyset$, the space $H_4(A)$ is the Sorgenfrey line which, as usual, is denoted by $S$. If $A_1 \cup A_2 \cup A_3 = \emptyset$, we denote $H(A)$ by $S^\leftarrow$. Of course, the spaces $S$ and $S^\leftarrow$ are homeomorphic.

The set-theoretic framework for this work is the Zermelo-Fraenkel system $ZF$. We recall that the Axiom of Choice (AC) is not an axiom of ZF; in consequence, our reasoning in ZF may require more subtle arguments than in ZFC and, moreover, some known ZFC-theorems on generalized ordered spaces, so extensively studied by many authors, cannot be applied to our work because they may fail in ZF. To the best of our knowledge, this article is the first in which hybrid topologies on $\mathbb{R}$ (among them the significant Hattori spaces) are investigated from the ZF-viewpoint. We believe that this feature and our results will attract the interest of topologists and set theorists.

The main aim of this article is to give a quasi-metrization theorem for hybrids of type $H_4(A)$ in the absence of the Axiom of Choice (see Theorems 3.2, 3.4 and Corollary 3.5). We also prove that Kofner’s quasi-metrization theorem (see [21, Theorem 10]) is false if there exists an infinite Dedekind-finite subset of the real line (see Definition 8 and Theorem 3.11). We also give a sufficient condition for $H_4(A)$ to be metrizable in ZF (see Propositions 3.13–3.14). Finally, by showing in Proposition 3.15 that all spaces of type $H_4(A)$ are both normal and completely regular in ZF, we strengthen Theorem 2.3 of [26] asserting that every Hattori space is normal in ZFC.

In Section 2, we establish set-theoretic and topological notation and terminology. Section 3 contains our main results. Open problems are listed in Section 4.
2. Notation and terminology

In this paper, we use standard set-theoretic notation and terminology. To avoid misunderstanding, let us recall the following well-known concepts.

Definition 1. A set $X$ is *Dedekind-finite* if there does not exist a proper subset of $X$ equipotent to $X$; otherwise, $X$ is *Dedekind-infinite*.

As usual, $\omega$ stands for the set of all Dedekind-finite ordinal numbers (of von Neumann).

Definition 2. A set $X$ is:

(i) finite if $X$ is equipotent to an element of $\omega$; otherwise, $X$ is *infinite*;

(ii) countable if $X$ is equipotent to a subset of $\omega$;

(iii) infinitely countable or, equivalently, *denumerable* if $X$ is equipotent to $\omega$.

It is well known that it is true in ZF that a set $X$ is Dedekind-infinite if and only if it has a denumerable subset. The notions of a Dedekind-infinite set and an infinite set are equivalent in ZFC but not in ZF.

Remark. Form 13 of [17] is the statement: “Every Dedekind-finite subset of $\mathbb{R}$ is finite”. In Cohen’s original model $M_1$ of [17], there exists an infinite Dedekind-finite subset of $\mathbb{R}$. Basic properties of infinite Dedekind-finite subsets of $\mathbb{R}$ can be found in [14].

Form 9 of [17] is the statement: “Every Dedekind-finite set is finite”. It is known that there exists a model $\mathcal{M}$ of ZF in which every linearly ordered Dedekind-finite set is finite (thus, Form 13 of [17] is true in $\mathcal{M}$) but Form 9 of [17] is false in $\mathcal{M}$ (see [17, p. 324]). Therefore, Forms 9 and 13 of [17] are not equivalent in ZF. Form 9 of [17] is essentially stronger than Form 13 of [17].

In the sequel, for our convenience, we denote by $\mathbb{N}$ the set $\omega \setminus \{0\}$ where $0 = \emptyset$. For every ordinal $\alpha$, $\alpha + 1 = \alpha \cup \{\alpha\}$. We treat $\omega$ as the set of all non-negative integers in $\mathbb{R}$. The set of all rational numbers is denoted by $\mathbb{Q}$.

Given a topological space $X$ with the underlying set $X$, we denote by $\tau[X]$ the topology of $X$. For a set $A \subseteq X$, $\operatorname{cl}_X(A)$ and $\operatorname{cl}_{\tau[X]}(A)$ denote the closure of $A$ in $X$.

Usually, if it is not stated otherwise, we denote topological spaces with boldface letters and their underlying sets with lightface letters. We may also denote the topology of $X$ by $\tau$, writing $X = (X, \tau)$.

In particular, the topology of the Sorgenfrey line is denoted by $\tau[S]$. The natural topology of the real line $\mathbb{R}$, determined by the standard absolute value on
\( \mathbb{R} \), is denoted by \( \tau[\mathbb{R}] \). For simplicity, if this is not confusing, we call the space \( \langle \mathbb{R}, \tau[\mathbb{R}] \rangle \) simply the real line and denote it by \( \mathbb{R} \).

Let us recall some basic definitions concerning quasi-metrics to discuss, in the forthcoming Section 3, conditions on \( A \) to get the quasi-metrizability of \( H_4(\mathcal{A}) \).

**Definition 3.** (Cf. [9, pp. 3, 129], [12, Definition 1.10, p. 488], [30].) A quasi-metric (respectively, non-archimedean quasi-metric) on a set \( X \) is a function \( \rho : X \times X \to [0, +\infty) \) satisfying the following conditions:

(i) \( (\forall x, y \in X)(\rho(x, y) = 0 \iff x = y) \);

(ii) \( (\forall x, y, z \in X)\rho(x, y) \leq \rho(x, z) + \rho(z, y) \)

(respectively, \( (\forall x, y, z \in X)\rho(x, y) \leq \max\{\rho(x, z), \rho(z, y)\} \)).

Given a quasi-metric \( \rho \) on a set \( X \), a point \( x \in X \) and a real number \( r > 0 \), we put \( B_\rho(x, r) = \{y \in X : \rho(x, y) < r\} \). We denote by \( \tau(\rho) \) the topology on \( X \) having the collection \( \{B_\rho(x, r) : x \in X \land r > 0\} \) as a base. We call \( \tau(\rho) \) the topology induced by \( \rho \). The quasi-metric \( \rho \) is a metric if it satisfies the condition:

\( (\forall x, y \in X)\rho(x, y) = \rho(y, x) \). For a non-empty subset \( A \) of \( X \) and a point \( x \in X \), \( \rho(x, A) = \inf\{\rho(x, y) : y \in A\} \).

**Definition 4.** (Cf. [9, pp. 3, 129], [12, Definition 10.1, p. 488], [30].)

(i) A topological space \( \langle X, \tau \rangle \) is called (non-archimedeanly) quasi-metrizable if there exists a (non-archimedean) quasi-metric \( \rho \) on \( X \) such that \( \tau = \tau(\rho) \).

(ii) If \( \langle X, \tau \rangle \) is a quasi-metrizable space, then every quasi-metric \( \rho \) on \( X \) such that \( \tau = \tau(\rho) \) is called a quasi-metric for \( \langle X, \tau \rangle \).

Some authors use different definitions of a quasi-metric, not necessarily equivalent to Definition 3. For instance, in [26], condition (i) of Definition 3 is replaced with the following: \( (\forall x, y \in X)(\rho(x, y) = \rho(y, x) = 0 \iff x = y) \). However, in this paper, similarly to [9] and [12], we follow Wilson’s terminology from [30] and use the term “quasi-metric” only in the sense of Definition 3.

**Definition 5.** (Cf. [9, pp. 29], [12, p. 489].) A family \( \mathcal{U} \) of open subsets of a topological space \( X \) is called:

(i) interior-preserving if, for every non-empty subfamily \( \mathcal{V} \) of \( \mathcal{U} \), the intersection \( \bigcap \mathcal{V} \) is open in \( X \);

(ii) \( \sigma \)-interior-preserving if there exists a family \( \{\mathcal{U}_n : n \in \omega\} \) such that \( \mathcal{U} = \bigcup_{n \in \omega} \mathcal{U}_n \) and, for every \( n \in \omega \), \( \mathcal{U}_n \) is interior-preserving in \( X \).

If \( d \) is a metric on a set \( X \) and \( A \) is a non-empty subset of \( X \), then \( \text{diam}_d(A) = \sup\{d(x, y) : x, y \in A\} \).
Definition 6. Let $X = (X, d)$ be a metric space. Then $d$ is called a Cantor complete metric if, for every family $\{F_n : n \in \omega\}$ of non-empty closed sets in $X$ such that $\lim_{n \to +\infty} \text{diam}_d(F_n) = 0$ and, for every $n \in \omega$, $F_{n+1} \subseteq F_n$, it holds that $\bigcap_{n \in \omega} F_n \neq \emptyset$.

A topological space $Y$ is called a Cantor completely metrizable space if its topology is induced by a Cantor complete metric on $Y$.

Definition 7. A topological space $X$ is called a Loeb space if it is either empty or the family of all non-empty closed sets in $X$ has a choice function.

The concepts of a Cantor completely metrizable space and a Loeb space have been investigated in various articles by several authors; however, in this paper, we limit our attention only to some results relevant to these concepts which given in [18]–[20] and directly applied in the next section.

All topological notions used in this paper, if not introduced here, are standard and can be found in [7] or [29].

3. The main results

To begin with, let us notice that the following well-known theorem is provable in $\text{ZF}$.

Theorem 3.1. (Cf. [9, p. 151], [12, Theorem 10.3, p. 490].) $\text{ZF}$ A $T_1$-space is non-archimedeanly quasi-metrizable if and only if it has a $\sigma$-interior-preserving base.

In the sequel, we assume that $A = \{A_1, A_2, A_3, A_4\}$ is a given 4-cover of $\mathbb{R}$.

Now, we are in a position to state our first main theorem on $H_4(A)$.

Theorem 3.2. $\text{ZF}$ Suppose that the 4-cover $A$ satisfies the following conditions:

(i) there exists a family $\{F_n : n \in \omega\}$ of subsets of $A_3$ such that $A_3 = \bigcup_{n \in \omega} F_n$ and, for every $n \in \omega$, $\text{cl}_{S^-}(F_n) \subseteq A_2 \cup A_3$;

(ii) there exists a family $\{H_n : n \in \omega\}$ of subsets of $A_4$ such that $A_4 = \bigcup_{n \in \omega} H_n$ and, for every $n \in \omega$, $\text{cl}_{S}(H_n) \subseteq A_2 \cup A_4$.

Then the space $H_4(A)$ is non-archimedeanly quasi-metrizable. In particular, if $A_3$ is of type $F_\sigma$ in $S^-$, and $A_4$ is of type $F_\sigma$ in $S$, then $H_4(A)$ is non-archimedeanly quasi-metrizable.

Proof. In view of Theorem 3.1, it suffices to show a $\sigma$-interior-preserving base for $H_4(A)$.

We fix families $\{F_n : n \in \omega\}$ and $\{H_n : n \in \omega\}$ which have the properties described in (i) and (ii), respectively.
Let $\mathcal{U}^d = \{\{x\} : x \in A_2\}$. Of course, $\mathcal{U}^d$ is interior-preserving in $H_4(\mathcal{A})$.

Given $n \in \omega$ and $q \in \mathbb{Q}$, we put $\mathcal{U}_q(n) = \{\{x, q\} : x \in F_n \land x < q\}$ and $\mathcal{V}_q(n) = \{(q, x) : x \in H_n \land q < x\}$.

Let us show that the family $\mathcal{V}_q(n)$ is interior-preserving in $H_4(\mathcal{A})$. To this end, we fix a non-empty subfamily $\mathcal{W}$ of $\mathcal{V}_q(n)$. We may assume that $\bigcap \mathcal{W} \neq \emptyset$. We fix a point $x_0 \in \bigcap \mathcal{W}$ and put $\mathcal{V}_q(n, x_0) = \{V \in \mathcal{V}_q(n) : x_0 \in V\}$. Since $x_0 \in \bigcap \mathcal{W}$, we have $\mathcal{V}_q(n, x_0) \neq \emptyset$. Let $V(x_0) = \bigcap \mathcal{V}_q(n, x_0)$. Of course, $x_0 \in V(x_0) \subseteq \bigcap \mathcal{W}$.

To prove that $\mathcal{V}_q(n)$ is interior-preserving in $H_4(\mathcal{A})$, it suffices to check that $V(x_0)$ is open in $H_4(\mathcal{A})$.

Let $Y(x_0) = \{x \in H_n : x_0 \leq x\}$. Since $\mathcal{V}_q(n, x_0) \neq \emptyset$, we infer that $Y(x_0) \neq \emptyset$. Let $y_0 = \inf Y(x_0)$. Then $q < x_0 \leq y_0$ and $(q, y_0] \subseteq V(x_0) \subseteq \bigcap \{(q, y] : y \in Y(x_0)\} \subseteq (q, y_0]$. Therefore, $V(x_0) = (q, y_0]$. Since $y_0 \in \text{cl}_\tau(H_n) \subseteq A_4 \cup A_2$, it follows that $(q, y_0] \in \tau[H_4(\mathcal{A})]$. This proves that all the families $\mathcal{V}_q(n)$ are interior-preserving in $H_4(\mathcal{A})$. Using similar arguments, one can show that, for every $q \in \mathbb{Q}$ and every $n \in \omega$, the family $\mathcal{U}_q(n)$ is also interior-preserving in $H_4(\mathcal{A})$.

For every pair $p, q$ of rational numbers with $p < q$, we put $\mathcal{W}_{p, q} = \{(p, q)\}$. The family $\mathcal{W}_{p, q}$ is interior-preserving in $H_4(\mathcal{A})$. Furthermore, the union $\mathcal{U}^d \cup \bigcup \{\mathcal{W}_{p, q} : p, q \in \mathbb{Q}, p < q\} \cup \bigcup \{\mathcal{U}_q(n) \cup \mathcal{V}_q(n) : q \in \mathbb{Q} \land n \in \omega\}$ is a $\sigma$-interior-preserving base for $H_4(\mathcal{A})$. \hfill $\square$

**Corollary 3.3.** [ZF] Suppose that the $4$-cover $\mathcal{A}$ is such that either $A_1 \cup A_3 = \emptyset$ or $A_1 \cup A_4 = \emptyset$, or $A_3 \cup A_4 = \emptyset$. Then the hybrid space $H_4(\mathcal{A})$ is non-archimedeanquasi-metrizable.

Under the assumptions of Corollary 3.3, a quasi-metric (in general, not non-archimedean) for $H_4(\mathcal{A})$ is shown in [26, Theorem 5.2].

One may try to deduce our Theorem 3.2 from [21, Theorem 10] or from [1, Theorem 3.1]; however, the results of [1] and [21] were proved in ZFC but not in ZF, no detailed ZFC-proof of [21, Theorem 10] is included in [21], and the original proof of [1, Theorem 3.1] in a more general setting is too complicated for ZF-proofs of our theorems. Furthermore, we shall show later that [21, Theorem 10] is unprovable in ZF. This is partly why we have given a simple, detailed proof of Theorem 3.2 and we give below a very clarified proof of Theorem 3.4 stating that the assumptions of Theorem 3.2 are also necessary for $H_4(\mathcal{A})$ to be quasi-metrizable in ZF.

**Theorem 3.4.** [ZF] If $H_4(\mathcal{A})$ is quasi-metrizable, then $\mathcal{A}$ satisfies conditions (i) and (ii) of Theorem 3.2.
this aim, we fix a bijection $q : \omega \to \mathbb{Q}$. For every $k \in \omega$, let $Q_k = \{q(i) : i \in k + 1\}$. For every $n \in \omega$ and every $x \in \mathbb{R}$, let $E_n(x) = B_{\rho}(x, \frac{1}{2^n})$. For $x \in A_3$ and $n \in \omega$, let $m(x, n) = \min\{i \in \omega : [x, x + \frac{1}{2^n}) \subseteq E_n(x)\}$, $j(x, n) = \max\{n, m(x, n)\}$ and $D_n(x) = [x, x + \frac{1}{2^{m(x, n)}})$. For every pair $k, n \in \omega$, we define a subset $F_{k,n}$ of $A_3$ as follows:

$$F_{k,n} = \{x \in A_3 : E_n(x) \subseteq [x, +\infty) \land D_{n+1}(x) \cap (Q_k \setminus \{x\}) \neq \emptyset\}.$$  

It is obvious that $A_3 = \bigcup\{F_{k,n} : k, n \in \omega\}$. Let us fix $k, n \in \omega$ and show that $\text{cl}_{\text{p}}(F_{k,n}) \subseteq A_2 \cup A_3$.

Let $p \in \text{cl}_{\text{p}}(F_{k,n})$. Suppose that $p \in A_1 \cup A_4$. Let $q_0 = \max Q_k$. Suppose that $q_0 < p$. Then there exists $x_0 \in F_{k,n} \cap (q_0, p]$. But this is impossible because $D_{n+1}(x_0) \cap Q_k \neq \emptyset$. The contradiction obtained shows that $p \leq q_0$. Let $q_p = \min\{q(i) : i \in k + 1 \land p \leq q(i)\}$. We can choose $\epsilon > 0$ such that $(p - \epsilon, p) \cap Q_k = \emptyset$. We notice that if $x \in F_{k,n} \cap (p - \epsilon, p)$, then $q_p \in D_{n+1}(x)$, so $p \in D_{n+1}(x)$.

Since $p \in \text{cl}_{\text{p}}(F_{k,n})$ and $p \in A_1 \cup A_4$, we deduce that there exist $x_1, x_2 \in F_{k,n} \cap E_{n+1}(p) \cap (p - \epsilon, p)$ such that $x_1 < x_2$. Since $p \in D_{n+1}(x_2) \subseteq E_{n+1}(x_2)$, we infer that $E_{n+1}(p) \subseteq E_n(x_2)$. Hence $x_1 \in E_n(x_2) \subseteq [x_2, +\infty)$. This is impossible. The contradiction obtained proves that $p \in A_0 \cup A_3$. Hence (i) of Theorem 3.2 is satisfied. Arguing similarly, one can show that also condition (ii) of Theorem 3.2 is satisfied.

**Corollary 3.5.** [\text{ZF}] For every 4-cover $\mathcal{A}$ of $\mathbb{R}$, the following conditions are all equivalent:

(i) $H_4(\mathcal{A})$ is non-archimedeanly quasi-metrizable;
(ii) $H_4(\mathcal{A})$ is quasi-metrizable;
(iii) $\mathcal{A}$ satisfies conditions (i)–(ii) of Theorem 3.2.

**Corollary 3.6.** [\text{ZF}] Suppose that the 4-cover $\mathcal{A}$ is such that $A_2 = \emptyset$. Then the space $H_4(\mathcal{A})$ is quasi-metrizable if and only if $A_3$ is of type $F_\sigma$ in $\mathcal{S}^+$, and $A_4$ is of type $F_\sigma$ in $\mathcal{S}$.

**Problem.** Find, if possible, a subset $F$ of $\mathbb{R}$ such that $F$ is of type $F_\sigma$ in $\mathcal{S}$ but $F$ is not of type $F_\sigma$ in $\mathbb{R}$.

To show our next corollary and that Theorem 10 of [21] is unprovable in \text{ZF}, we need the following lemma. We include its \text{ZF}-proof for completeness.

**Lemma 3.7.** [\text{ZF}] If a subset $F$ of $\mathbb{R}$ is closed either in $\mathcal{S}$ or in $\mathcal{S}^+$, then $F$ is of type $G_\delta$ in $\mathbb{R}$.
PROOF. Suppose that $F \subseteq \mathbb{R}$ is closed in $\mathbb{S}^{-}$. We put $H = \text{cl}_{\mathbb{R}}(F)$ and $C = H \setminus F$. Since $F$ is closed in $\mathbb{S}^{-}$, for every $x \in C$, the set $J(x) = \{j \in \mathbb{N} : (x - \frac{1}{j}, x] \cap F = \emptyset\}$ is non-empty, so we can define $j(x) = \min J(x)$. Since the family $\{(x - \frac{1}{j(x)}, x] : x \in C\}$ is disjoint, it is countable. Therefore, the set $C$ is countable. This implies that $\mathbb{R} \setminus C$ is of type $G_{\delta}$ in $\mathbb{R}$. Since $H$ is also of type $G_{\delta}$ in $\mathbb{R}$ (in fact, in $\text{ZF}$, a closed subset of any metrizable space is of type $G_{\delta}$), the set $F = H \cap (\mathbb{R} \setminus C)$ is of type $G_{\delta}$ in $\mathbb{R}$. To complete the proof, we consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by: for every $x \in \mathbb{R}$, $f(x) = -x$. Since $f$ is a homeomorphism of $\mathbb{S}$ onto $\mathbb{S}^{\sim}$, we infer that every closed set of $\mathbb{S}$ is also of type $G_{\delta}$ in $\mathbb{R}$.

Corollary 3.8. [ZF] Assume that the 4-cover $A$ is such that $A_2$ is of type $F_\sigma$ in $\mathbb{R}$ but either $A_3$ or $A_4$ is not of type $G_{\delta \sigma}$ in $\mathbb{R}$. Then the hybrid space $H_4(A)$ is not quasi-metrizable.

PROOF. Suppose the space $H_4(A)$ is quasi-metrizable. It follows from Theorem 3.4 that there exists a family $\{F_n : n \in \omega\}$ of subsets of $A_3$ such that $A_3 = \bigcup_{n \in \omega} F_n$ and, for every $n \in \omega$, $\text{cl}_{\mathbb{S}^{\sim}}(F_n) \subseteq A_3 \cup A_2$. We fix $n \in \omega$ and put $H_n = \text{cl}_{\mathbb{S}^{\sim}}(F_n)$. By Lemma 3.7, the set $H_n$ is of type $G_{\delta}$ in $\mathbb{R}$. Since $A_2$ is of type $F_\sigma$ in $\mathbb{R}$, the set $E_n = H_n \setminus A_2$ is of type $G_{\delta}$ in $\mathbb{R}$. Clearly, $F_n \subseteq E_n \subseteq A_3$. Hence $A_3 = \bigcup_{n \in \omega} E_n$ is of type $G_{\delta \sigma}$ in $\mathbb{R}$. Using similar arguments, one can also prove that $A_4$ is of type $G_{\delta \sigma}$ in $\mathbb{R}$. This completes the proof.

We are going to show below that Kofner’s quasi-metrization theorem (see [21, Theorem 10]) may fail in $\text{ZF}$. To do this, in the following definition, we translate condition (ii) of [21, Theorem 10] into our language for Hattori spaces.

Definition 8. Kofner’s quasi-metrization theorem for Hattori spaces is the following statement: For every subset $A$ of $\mathbb{R}$, the Hattori space $H(A)$ is non-archimedeanly quasi-metrizable if and only if there exists a family $\{R_n : n \in \omega\}$ of subsets of $\mathbb{R}$ such that $\mathbb{R} \setminus A = \bigcup_{n \in \omega} R_n$ and, for every $n \in \omega$, $R_n$ is closed under limits in $\mathbb{R}$ of increasing sequences. (See [21, (ii) $\leftrightarrow$ (iii) of Theorem 10].)

Remark. Let $A \subseteq \mathbb{R}$. It follows from Corollary 3.6 that the following theorem is true in $\text{ZF}$: if the Hattori space $H(A)$ is quasi-metrizable, then there exists a family $\{R_n : n \in \omega\}$ such that $\mathbb{R} \setminus A = \bigcup_{n \in \omega} R_n$ and, for every $n \in \omega$, $R_n$ is closed under limits of increasing sequences. That the converse may fail in $\text{ZF}$ is shown in the forthcoming Theorem 3.11.

We need to apply to the proof Theorem 3.11 the following theorem which easily follows from some results of [19] and [20]; however, we outline its proof for the convenience of readers and to clarify some reasonings from [20].
Theorem 3.9. [ZF] If X is a Cantor completely metrizable second-countable space, then the following conditions are all satisfied:

(i) every $G_\delta$-subspace of X is homeomorphic with a closed subspace of $X \times \mathbb{R}^\omega$;
(ii) every $G_\delta$-subspace and every $F_\sigma$-subspace of X is Loeb;
(iii) every $G_\delta$-subspace and every $F_\sigma$-subspace of X is separable.

Proof. Let $\rho$ be a Cantor complete metric on set X and let $X = \langle X, \tau(\rho) \rangle$. Suppose that $\{G_n : n \in \omega\}$ is a family of open sets in X and $G = \bigcap_{n \in \omega} G_n \neq X$. We may assume that, for every $n \in \omega$, $G_n \neq X$.

(i) Although it was observed in the proof of Theorem 2.7 in [20] that (i) holds, let us give a little more detailed proof here. Similarly to the proof of Lemma 4.3.22 in [7], we define a mapping $h : G \to X \times \mathbb{R}^\omega$ as follows: for every $x \in G$, $h(x) = \langle x, t \rangle$ where $t \in \mathbb{R}^\omega$ is such that, for every $n \in \omega$, $t(n) = \rho(x, X \setminus G_n)$. The mapping $h$ is a homeomorphic embedding of $G$ onto the closed subspace $h[G]$ of $X \times \mathbb{R}^\omega$.

(ii) Now, we show that the subspace $G$ of X is Loeb. As in [20, proof of Theorem 2.7], we recall that it was shown in [18] that countable products of Cantor completely metrizable spaces are Cantor completely metrizable. Hence, the space $X \times \mathbb{R}^\omega$ is Cantor completely metrizable. It is also second-countable. By [19, Theorem 2.3], every Cantor completely metrizable second-countable space is Loeb. Of course, every closed subspace of a Loeb space is also Loeb. In consequence, the closed subspace $h[G]$ of $X \times \mathbb{R}^\omega$ is Loeb, so the subspace $G$ of X is Loeb.

We have already observed that the space X is Loeb. By [20, Proposition 2.1], every $F_\sigma$-subspace of a Loeb space is Loeb. Hence, every $F_\sigma$-subspace of X is Loeb.

(iii) It is known from [20, Lemma 2.9] and easy to prove that every second-countable regular Loeb space is separable. This, together with (ii), implies that (iii) holds.

The following corollary strengthens Corollary 2.11 of [20].

Corollary 3.10. [ZF]. Let D be an infinite subset of a Cantor completely metrizable second-countable space. If D is either of type $G_\delta$ or of type $F_\sigma$ in X, then D is Dedekind-infinite.

Proof. Let $\{D_n : n \in \omega\}$ be a family of $G_\delta$-sets in X such that $D = \bigcup_{n \in \omega} D_n$. It follows from Theorem 3.9(iii) that, for every $n \in \omega$, the subspace $D_n$ of X is separable. Hence, if there exists $n_0 \in \omega$ such that the set $D_{n_0}$ is infinite, then $D_{n_0}$ (so also D) contains a countably infinite subset. On the other hand, if the
sets $D_n$ are all finite, the set $D$ is of type $F_\sigma$ in $X$ and, by Theorem 3.9(iii), the subspace $D$ of $X$ is separable. In consequence, $D$ contains a countably infinite subset, so $D$ is Dedekind-infinite. □

We recall that, in [25], A. Miller constructed a model of $\text{ZF}$ in which there exists an infinite Dedekind-finite subset of $\mathbb{R}$ of type $F_{\sigma\delta}$ in $\mathbb{R}$. Therefore, an infinite $F_{\sigma\delta}$-subspace (also a $G_{\delta\sigma\delta}$-subspace) of a Cantor completely metrizable second-countable space may fail to be Loeb and can be Dedekind-finite in a model of $\text{ZF}$. The following interesting problem seems to be still unsolved:

**Problem.** Is it provable in $\text{ZF}$ that every Dedekind-finite subset of $\mathbb{R}$ is Borel in $\mathbb{R}$?

**Theorem 3.11.** ($\text{ZF}$) If $D$ is an infinite Dedekind-finite subset of $\mathbb{R}$, then the Hattori space $H(\mathbb{R} \setminus D)$ is not quasi-metrizable. In consequence, Kofner’s quasi-metrization theorem is false in every model of $\text{ZF}$ in which $\mathbb{R}$ contains an infinite Dedekind-finite set. In particular, Kofner’s theorem is false in Cohen’s original model $M_1$ in [17].

**Proof.** Let $D$ be an infinite subset of $\mathbb{R}$. Put $A = \mathbb{R} \setminus D$ and suppose that the Hattori space $H(A)$ is quasi-metrizable. Then it follows from Corollary 3.8 that the set $D$ is of type $G_{\delta\sigma}$ in $\mathbb{R}$. Since $\mathbb{R}$ is a Cantor completely metrizable second-countable space, we deduce from Corollary 3.10 that the set $D$ is Dedekind-infinite. □

It seems that a hard problem is to find useful necessary and sufficient conditions for $\mathcal{A}$ to determine a metrizable $H_4(\mathcal{A})$. Before we give a partial solution to this problem, let us make comments on the well-known Nagata-Smirnov metrization theorem, that is, on the statement: “A $T_3$-space is metrizable if and only if it has a $\sigma$-locally finite base”. The Nagata-Smirnov metrization theorem has an elegant proof in $\text{ZFC}$ (see, e.g., [7, proof of Theorem 4.4.7]).

Let us denote by $\text{MP}$ the statement: “Every metric space is paracompact”. This statement is Form 383 in [17]. It was A. H. Stone who proved that $\text{MP}$ is true in $\text{ZFC}$ (see [28, Corollary 1]). The fact that $\text{MP}$ is not provable in $\text{ZF}$ was first established in [11]. Form 232B of [17] is the statement: “Every metric space has a $\sigma$-locally finite base”. That Form 232B of [17] implies $\text{MP}$ in $\text{ZF}$ was established in [16]. That the reverse implication is also true in $\text{ZF}$ was established in [15, Theorem 14]. In consequence, the Nagata-Smirnov metrization theorem is unprovable in $\text{ZF}$. However, it was observed in [6] that, in view of [5, proof of Theorem 7], the following part of the Nagata-Smirnov metrization theorem is true in $\text{ZF}$.
Theorem 3.12. [ZF] If a $T_3$-space has a $\sigma$-locally finite base, it is metrizable.

Now, it is easily seen that the Nagata-Smirnov metrization theorem is equivalent to MP in ZF. With Theorem 3.12 in hand, we can state the following simple proposition.

Proposition 3.13. [ZF] If the 4-cover $A$ is such that $A_2$ is of type $F_\sigma$ in $\mathbb{R}$, and $A_3 \cup A_4$ is countable, then the hybrid space $H_4(A)$ is metrizable.

Proof. Under the assumptions of the proposition, the space $H_4(A)$ has a $\sigma$-locally finite base, so it is metrizable by Theorem 3.12. $\square$

Proposition 3.14. [ZF] Suppose that the 4-cover $A$ is such that $A_2$ is countable. Then the following conditions are equivalent:

(i) $H_4(A)$ is second-countable;
(ii) $H_4(A)$ is metrizable;
(iii) $A_3 \cup A_4$ is countable.

Proof. Since $A_2$ is countable, the space $H_4(A)$ is separable. Hence, if $H_4(A)$ is metrizable, it is also second-countable. Clearly, a subspace of the Sorgenfrey line is second-countable if and only if it is countable. Every second-countable $T_3$-space is metrizable by Theorem 3.12, and, moreover, $A_2 \cup A_3 \cup A_4$ is countable if and only if $H_4(A)$ is second-countable. All this taken together completes the proof. $\square$

Let us recall that even a linearly ordered normal space can fail to be completely regular in ZF (see, e.g., [10]), and a linearly orderable topological space may fail to be normal in ZF (see [17, Form 118, p. 42]). In [26], the proof that every Hattori space is $T_4$ (see [26, Theorem 2.3]) was conducted in ZFC; however, its minor modifications yield a ZF-proof of [26, Theorem 2.3]. The following proposition strengthens [26, Theorem 2.3].

Proposition 3.15. [ZF] For every 4-cover $A$ of $\mathbb{R}$, the space $H_4(A)$ is both normal and completely regular.

Proof. Let $C_0, C_1$ be a pair of non-empty disjoint closed sets of $H_4(A)$. Let $i \in \{0, 1\}$ and $c \in C_i$. If $c \in A_2$, we put $U_i(c) = \{c\}$. Assuming that $c \in C_i \setminus A_2$, we define the set $N(c)$ as follows:

\[
N(c) = \begin{cases} 
\{n \in \mathbb{N} : (c - \frac{1}{n}, c + \frac{1}{n}) \cap C_{1-i} = \emptyset\} & \text{if } c \in A_1; \\
\{n \in \mathbb{N} : [c, c + \frac{1}{n}) \cap C_{1-i} = \emptyset\} & \text{if } c \in A_3; \\
\{n \in \mathbb{N} : (c - \frac{1}{n}, c] \cap C_{1-i} = \emptyset\} & \text{if } c \in A_4.
\end{cases}
\]
We put \( n(c) = \min N(c) \) and define the set \( U_i(c) \) as follows:

\[
U_i(c) = \begin{cases} 
(c - \frac{1}{2n(c)}, c + \frac{1}{2n(c)}) & \text{if } c \in A_1; \\
[c, c + \frac{1}{2n(c)}) & \text{if } c \in A_3; \\
(c - \frac{1}{2n(c)}, c) & \text{if } c \in A_4.
\end{cases}
\]

We define \( V_i = \bigcup \{ U_i(c) : c \in C_i \} \). Clearly, \( V_i \in \tau[H_4(A)] \) and \( C_i \subseteq V_i \). Arguing in much the same way as in the proof of Theorem 2.3 in [26], one can check that \( V_0 \cap V_1 = \emptyset \). This proves that the space \( H_4(A) \) is normal.

To show that the space \( H_4(A) \) is completely regular, we consider any closed set \( E \) in \( H_4(A) \) and any point \( x \in \mathbb{R} \setminus E \). We choose \( U \in \mathcal{B}(x) \) such that \( U \cap E = \emptyset \). Suppose that \( x \in A_3 \). Then, for some \( \epsilon > 0, U = [x, x + \epsilon) \). We define the function \( f_x : \mathbb{R} \rightarrow [0, 1] \) as follows:

\[
f_x(t) = \begin{cases} 
1 & \text{if } t \in (-\infty, x) \cup [x + \epsilon, +\infty); \\
0 & \text{if } t \in [x, x + \frac{\epsilon}{2}]; \\
\frac{2}{\epsilon}(t - x) - 1 & \text{if } t \in (x + \frac{\epsilon}{2}, x + \epsilon).
\end{cases}
\]

Then \( f_x \) is a continuous function from \( H_4(A) \) into \([0, 1]\) such that \( f_x(x) = 0 \) and \( E \subseteq f_x^{-1}\{1\} \). Using similar arguments, one can show that if \( x \in A_1 \cup A_2 \cup A_4 \), then there exists a continuous function \( f_x \) from \( H_4(A) \) into \([0, 1]\) such that \( f_x(x) = 0 \) and \( E \subseteq f_x^{-1}\{1\} \). Hence, \( H_4(A) \) is completely regular. \( \square \)

4. Open problems and directions for future research

We would like to encourage readers to investigate other topological properties of the hybrid spaces \( H_4(A) \) in \( \text{ZF} \). For the convenience of readers we repeat below the open problems posed in Section 3 and add also new ones to show desirable directions for future research in this field. Clearly, many other open problems could be added.

1. Find, if possible, a subset \( F \) of \( \mathbb{R} \) such that \( F \) is of type \( F_\sigma \) in \( S \) but \( F \) is not of type \( F_\sigma \) in \( \mathbb{R} \).
2. Is it provable in \( \text{ZF} \) that every Dedekind-finite subset of \( \mathbb{R} \) is Borel in \( \mathbb{R} \)?
3. Find useful necessary and sufficient conditions for a 4-cover \( A \) of \( \mathbb{R} \) to determine a metrizable \( H_4(A) \).
4. Check which of the already published theorems on Hattori spaces, shown provable in \( \text{ZFC} \), may fail in \( \text{ZF} \).
5. Verify carefully which of the known from literature theorems on generalized ordered spaces in \( \text{ZFC} \) may fail in \( \text{ZF} \).
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