The Statistics of Fields

Outline

- Introduction
- Review of Regression
- Trend Surface Analysis: Regression on Spatial Coordinates
- Statistical Interpolation: Kriging

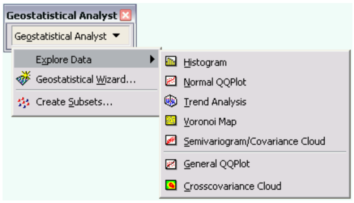
Introduction

- Previous interpolation methods all use specific mathematical functions (deterministic interpolation)
- **Problems:**
  - No environmental measurements can be made without error. It is ill-advised to try to honor all the observed data without recognizing the inherent variability
  - Deterministic methods assume that we know about how the variable being interpolated behaves spatially. However, the observed control point data may provide useful information that suggests otherwise.

Introduction

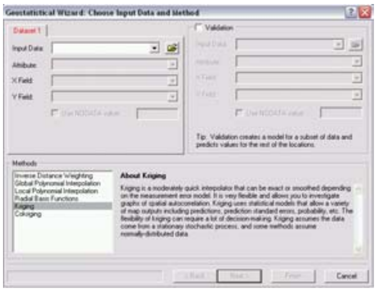
- Statistical (stochastic) interpolation
  - Trend surface analysis: specified functions are fitted to the locational coordinates (x,y) of the control point data in an attempt to approximate universal trends in field height (z): first order effect
  - Kriging: attempts to make optimum use of the underlying phenomenon as a spatially continuous field of non-independent random variables: second order effect

Introduction



Surface Trend Analysis (Geostatistical Analyst Extension)

Introduction



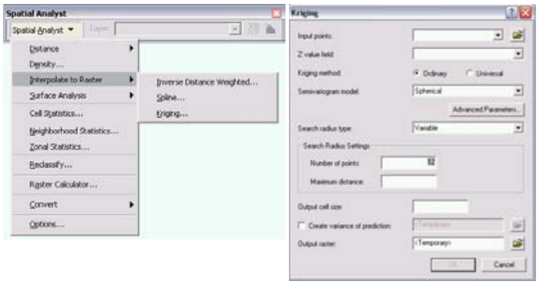
Tip: Validation creates a model for a subset of data and predicts values for the rest of the locations.

About Kriging

Kriging is a moderately quick interpolator that can be exact or smoothed depending on the measurement error model. It is very flexible and allows you to investigate graphs of spatial autocorrelation. Kriging uses statistical models that allow a variety of map outputs including predictions, prediction standard errors, probability, etc. The flexibility of kriging can require a lot of decision-making. Kriging assumes the data come from a stationary stochastic process, and some methods assume normally-distributed data.

Kriging (Geostatistical Analyst Extension)

Introduction



Input points:
Z value field:
Kriging method: ● Ordinary ○ Universal
Semivariogram model: Spherical
Search radius type: Variable
Number of points: 12
Maximum distance:
Output cell size:
☐ Create variance of prediction: <Temporary>
Output raster: <Temporary>

Kriging (Spatial Analyst Extension)

Introduction

Spatial interpolation (ArcToolbox)

Kriging

- Deterministic interpolation (e.g. local spatial average, IDW) determine the distance weighting function and neighborhood definition based on expert knowledge, not from the data
  - Based on a description of **spatial autocorrelation** given by a mathematical function
- Kriging estimates the choice of function, weights, & neighborhood from the sample data, and interpolate the data with these choices.
  - Based on a description of **spatial autocorrelation** given by sample data

Kriging

- Kriging is **optimal** in the sense that it makes **best** use of what can be inferred about the spatial structure in the surface to be interpolated from an analysis of the control point data
  - by David Krige for South African mining industry

Quick Reminder: Spatial Autocorrelation

- The relationship between spatial difference and attribute difference
  - Spatial difference measure: w_{ij}
  - Attribute difference measure: co-variance or variance

$$(z_i - \bar{z})(z_j - \bar{z}) \qquad (z_i - z_j)^2$$

Kriging

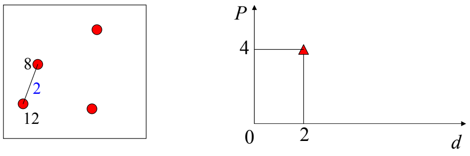
- **Three steps:**
  1) Produce a description of the spatial variation in the sample control point data
  2) Summarizing the spatial variation by a regular mathematical function
  3) Using this model to determine the interpolation weights

Kriging

- Describing the spatial variation: the semi-variogram

- Variogram cloud: a plot of a measure of height differences dz_{ij} (attribute difference) against the distance d_{ij} (spatial difference) between the control points for all possible pairs of points

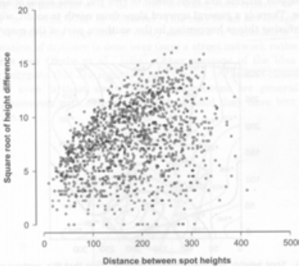
$$P(d_{ij}) = dz_{ij} = (z_i - z_j)^2$$



Kriging

- Describing the spatial variation: the semi-variogram

- Example of variogram cloud



There is a trend such that height differences increase as the separation distance increases

Indicates the farther apart two control points, the greater the likely difference in their value.

Kriging

- Describing the spatial variation: the semi-variogram

- Spatial dependence can be described more concisely by a **experimental** semi-variogram function as follows (often referred as variogram in short)

$$2\hat{\gamma}(d) = \frac{1}{n(d)} \sum_{d_{ij}=d} (z_i - z_j)^2$$

n(d) is the number of pair of points at separation d

$\hat{\gamma}$ is the estimated semi-variance

Kriging

- Describing the spatial variation: the semi-variogram

$$2\hat{\gamma}(d) = \frac{1}{n(d)} \sum_{d_{ij}=d} (z_i - z_j)^2$$

This is the theoretical equation for variogram estimation and it is not straightforward in applications

e.g. for a given distance d, it is more likely that there will be no pair of observations at precisely that separation.

Kriging

- Describing the spatial variation: the semi-variogram

In reality, variogram is estimated for different **bands** (or **lags**) rather than continuously at all distances.

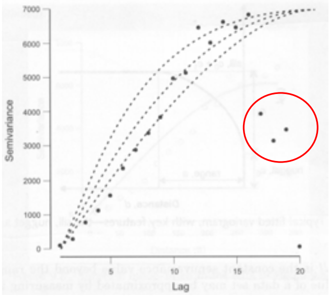
$$2\hat{\gamma}(d) = \frac{1}{n(d)} \sum_{d_{ij}=d-\Delta/2}^{d+\Delta/2} (z_i - z_j)^2$$

Δ is the lag width
$n(d)$ is the number of point pairs within (d- Δ/2, d+ Δ/2)

Kriging

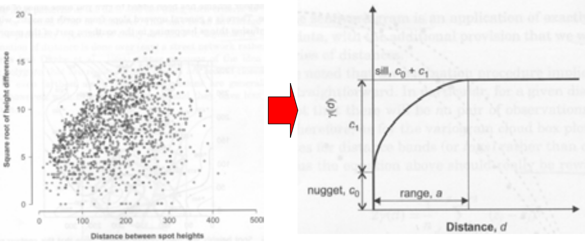
- Describing the spatial variation: the semi-variogram

To best use variogram, we need use a mathematical function to approximate the **detailed** information in the variogram clouds



Edge Effect

The Advantage of a Function

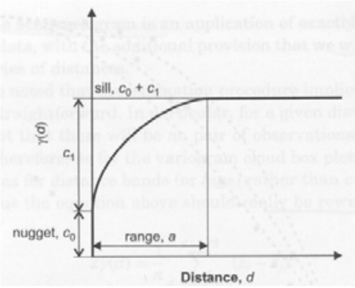


Exercise 19

- Ordinary Kriging
  - Step 1-7

Kriging

- Summarize the spatial variation by a regular mathematical function



Nugget (c_0): variance at zero distance

Range (a): the distance at which the semivariogram levels off and beyond which the semivariance is constant

Sill (c_0+c_1): the constant semivariance value beyond the range

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Kriging

- Summarize the spatial variation by a regular mathematical function

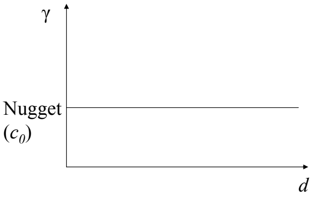
Mathematical Functions

- ✓Nugget model
- ✓Linear model
- ✓Spherical model
- ✓Exponential model
- ✓Power model
- ✓Gaussian model
- ✓Others

Kriging

- Summarize the spatial variation by a regular mathematical function

Nugget model: A constant variance model

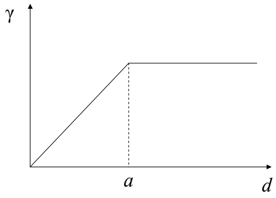


$\gamma = c_0$

Kriging

- Summarize the spatial variation by a regular mathematical function

Linear model: Variances change linearly with the change of distance

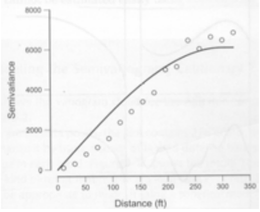


$\gamma = d$ When $d < a$

Kriging

- Summarize the spatial variation by a regular mathematical function

Spherical model starts from a nonzero variance (c_0) and rise as an elliptical arc to a maximum value (c_0+c_1) at distance a.



If $d <= a$ then

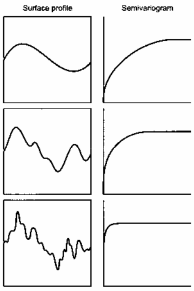
$$\gamma(d) = c_0 + c_1\left[\frac{3d}{2a} - 0.5\left(\frac{d}{a}\right)^3\right]$$

$$\gamma(d) = c_0 + c_1$$

Most often used!!!

Kriging

- Summarize the spatial variation by a regular mathematical function



Typical spatial profiles and their associated semivariograms

Exercise 19

- **Ordinary Kriging**
  - **Step 8**

Kriging

- Use the model to determine interpolation weights by Kriging

- In simple interpolation, unknown points is estimated as the *weighted* average of nearby points
- In Kriging, exactly the same, only that weights are estimated as well from semi-variogram and some nifty (incomprehensible) mathematics

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Kriging

- Use the model to determine interpolation weights by Kriging

Assumptions (Ordinary Kriging)

1) The surface has a constant mean, with no underlying trend
2) The surface is isotropic, having the same variation in each direction
3) The semivariogram is a simple mathematical model with some clearly defined properties
4) The same variogram applied over the entire area

Kriging

- Use the model to determine interpolation weights by Kriging

In Ordinary (ahem) Kriging, we need to solve:

$$\hat{z}_s = w_1 z_1 + w_2 z_2 + \ldots + w_n z_n$$

Minimize $E\{[\hat{z}_s - z_s]^2\}$

$$= 2\sum_{i=1}^{n} w_i \gamma(d_{is}) - \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \gamma(d_{ij})$$

Z: semi-variance estimated from the model function

Subject to: $w_1 + w_2 + \ldots + w_n = 1$

Kriging
- Use the model to determine interpolation weights by Kriging

- The important thing is that the weights which produces is different at each location (also true for other methods) and weights can be positive or negative, even greater than 1
  - Avoid “average effect”: value estimated may be outside the range of sample data

Kriging
- Use the model to determine interpolation weights by Kriging

Let the Software do this!!!!

ArcMap Geostatistics Extension
R Package
IDRISI (G-Stat)
GSLIB

Exercise 19

- Ordinary Kriging
  - Step 9-13

Kriging
- Use the model to determine interpolation weights by Kriging

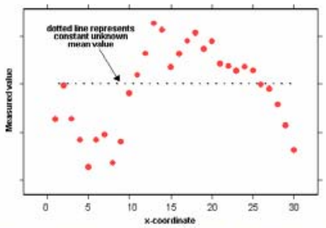
- **Conclusion**
  - Kriging is computationally intensive
  - All the results depend on the model we fit to the estimated semi-variogram from the sample data
  - IF the **corrected** model is used, the methods used in kriging have an advantage over other interpolation procedures

Kriging
- Use the model to determine interpolation weights by Kriging

- **Variations**
  - Simple kriging
  - Ordinary kriging (a constant mean)
  - Universal kriging (combine trend analysis with ordinary kriging)
  - Co-kriging (more than one variable)

Ordinary Kriging

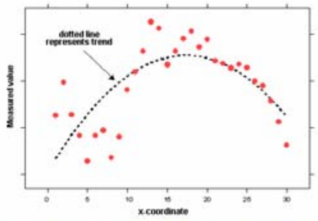
- Data contains a constant mean and no trend, the mean is unknown in advance



Ordinary Kriging assumes the data have a constant mean (no trend) and that the mean value is not known in advance. The data values (orange dots) are thought of as random errors that fluctuate around the unknown mean. The random errors are autocorrelated, meaning they tend to be above or below the mean in a way similar to their neighbors.

Universal Kriging

- Data contains a trend and the trend is unknown in advance



Universal Kriging assumes there is a trend in the data, but the terms of the trend function are not known in advance. The data values (orange dots) are thought of as random errors that fluctuate around the unknown trend. The random errors are autocorrelated, meaning they tend to be above or below the trend in a way similar to their neighbors.

Simple Kriging

- If the trend or constant mean is known in advance, then we have Simple Kriging.

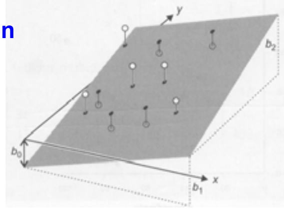
Trend Surface Analysis

- The trend of a surface is any large-scale **systematic** change that extends smoothly and predictably across the region of interest.
- It is an **exploratory** method to give a rough idea of the spatial pattern in a set of observations.
- Simply a multiple regression analysis:

$$z_i = f(s_i) = f(x_i, y_i)$$

Quick Reminder: Regression

- **Simple linear regression**



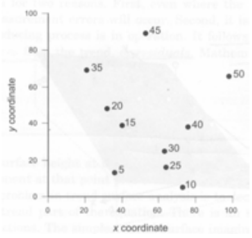
Dependent variable: z
Independent variable: x, y

$$z_i = b_0 + b_1 x_i + b_2 y_i + \varepsilon_i$$

$$\varepsilon_i = z_i - \bar{z}_i = z_i - b_0 - b_1 x_i - b_2 y_i$$

b_0 , b_1 & b_2 are coefficients to be estimated; ε_i is the error of estimation for i observation (**residual**)

Trend Surface Analysis



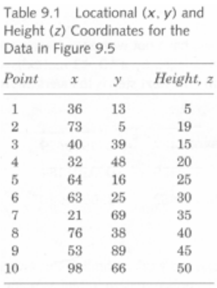


Table 9.1 Locational (x, y) and Height (z) Coordinates for the Data in Figure 9.5

Point	*x*	*y*	*Height, z*
1	36	13	5
2	73	5	19
3	40	39	15
4	32	48	20
5	64	16	25
6	63	25	30
7	21	69	35
8	76	38	40
9	53	89	45
10	98	66	50

Trend Surface Analysis

- **More complex surface**

Dependent variable: z
Independent variable: x, y

$$z_i = b_0 + b_1 x_i + b_2 y_i + b_3 x_i y_i + b_4 x_i^2 + b_5 y_i^2 + \varepsilon_i$$

b_0 , b_1, b_2 , b_3, b_4 & b_5 are coefficients to be estimated; ε_i is the error of estimation for i observation (**residual**)

Trend Surface Analysis

Problems

1. It is not reasonable to assume that the phenomenon of interest varies with the spatial coordinates in such a simple way
2. The fitted surfaces do not pass exactly through all the control points
3. Other than simple visualization of the pattern they appear to display, the data are not used to help select this model.

Exercise 19

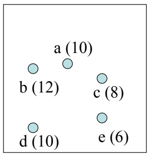
- Ordinary Kriging
  - The remaining steps

Summary

- Statistical interpolation methods
  - Trend Surface Analysis
  - Kriging

Kriging

- Describing the spatial variation: the semi-variogram



Distance matrix

$$\begin{array}{c|ccccc} & a & b & c & d & e \\ \hline a & 0 & 1 & 1 & 3 & 2.5 \\ b & 1 & 0 & 2 & 2 & 3 \\ c & 1 & 2 & 0 & 3 & 1 \\ d & 3 & 2 & 3 & 0 & 3 \\ e & 2.5 & 3 & 1 & 3 & 0 \end{array}$$

$\Delta = 0.5$
d = 0.5

What is the value of $\gamma(0.5)$?
What is the value of $\gamma(1.5)$?

Kriging

- Summarize the spatial variation by a regular mathematical function

Problems with variogram estimation

1. The reliability of the calculated semivariance varies with the number of point pairs used in their estimation
   - The lager, the more reliable
2. Spatial variation may be anisotropic (varies with directions), favoring change in a particular direction
3. It assumes there is no systematic spatial change in the mean surface height (first order effect)
4. The experimental semivariogram can fluctuate greatly from point to point
5. Many functions are non-linear

Kriging

- Use the model to determine interpolation weights by Kriging

Solve the above equation by using a *Lagrangian Multiplier*

$$w_1\gamma(d_{11}) + w_2\gamma(d_{12}) + \ldots + w_n\gamma(d_{1n}) + \lambda = \gamma(d_{1s})$$

$$\ldots\ldots$$

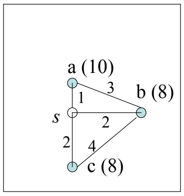
$$w_1\gamma(d_{n1}) + w_2\gamma(d_{n2}) + \ldots + w_n\gamma(d_{nn}) + \lambda = \gamma(d_{ns})$$

$$w_1 + w_2 + \ldots + w_n = 1$$

n+1 variables, *n+1* linear equations

Kriging

- Use the model to determine interpolation weights by Kriging



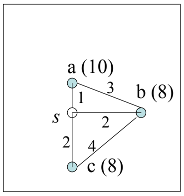
$\gamma(d) = 1$

$$w_a\gamma(d_{aa}) + w_b\gamma(d_{ab}) + w_c\gamma(d_{ac}) + \lambda = \gamma(d_{as})$$
$$w_a\gamma(d_{ba}) + w_b\gamma(d_{bb}) + w_c\gamma(d_{bc}) + \lambda = \gamma(d_{bs})$$
$$w_a\gamma(d_{ca}) + w_b\gamma(d_{cb}) + w_c\gamma(d_{cc}) + \lambda = \gamma(d_{cs})$$
$$w_a + w_b + w_c = 1$$

What is the value of w_a, w_b, w_c, and λ?
What is the value of s?

Kriging

- Use the model to determine interpolation weights by Kriging



$$\begin{cases} \gamma(d) = d & \text{if } d <= 2 \\ \gamma(d) = 2 & \text{if } d > 2 \end{cases}$$

What is the value of w_a, w_b, w_c, and λ?
What is the value of s?
What is the value of s with IDW method?

Kriging

- Summarize the spatial variation by a regular mathematical function

Variogram model fitting methods

1) Interactive model fitting
2) Weighted least squares (R and Gstat)
3) Modified weighted least squares (ArcMap Geostatistics)
4) Others

Kriging

- Use the model to determine interpolation weights by Kriging

Weights for **every** unknown location can be obtained by solving the above equation in use of a *Lagrangian Multiplier*

$$w_1\gamma(d_{11}) + w_2\gamma(d_{12}) + \ldots + w_n\gamma(d_{1n}) + \lambda = \gamma(d_{1s})$$

……

$$w_1\gamma(d_{n1}) + w_2\gamma(d_{n2}) + \ldots + w_n\gamma(d_{nn}) + \lambda = \gamma(d_{ns})$$
$$w_1 + w_2 + \ldots. + w_n = 1$$

n+1 variables, *n+1* linear equations