Article 00.: Introduction. Given a univariant, Quadratic polynomial with, say, real number coefficients,

\[ Q(x) = ax^2 + bx + c , \]

there is an associated monic, quadratic polynomial, \( M(x) \), which is described by the following,

**DEFINITION:** To each quadratic polynomial as in (0.1), there is an associated monic, quadratic polynomial, \( M(x) \), called the *monic adjunct of \( Q(x) \)*, and which is defined by,

\[ M(x) = x^2 + 2bx + 4ac \quad \text{(monic adjunct of \( Q \))}. \]

As a consequence of some casual, algebraic, gymnastic doodling, several functional relationships (and connections) regarding the zeros of \( Q(x) \) and \( M(x) \) did emerge. An account of such doodling--and the results there of--are below presented in the following sequel of Articles.

Article 01.: Monic (Quadratic) Adjunct Polynomials. Consider a quadratic, \( Q(x) \), as in Item (0.1). Now, note that

\[ 4a Q(x) = 4a^2 x^2 + 4abx + 4ac = (2ax)^2 + 2b(2ax) + 4ac . \]

So by declaring that \( z = 2ax \); and, that \( M(z) \) be defined as

\[ M(z) = z^2 + 2bz + 4ac , \]

the following relationships emerge.

\[ (1.3) \quad \boxed{\text{(A): } 4a Q(x) = M(2ax) \quad \text{and} \quad \text{(B): } Q\left(\frac{z}{2a}\right) = \frac{1}{4a} M(z)} . \]

Appealing to (1.2), further exploration in such manner reveals that (for \( z \neq 0 \))

\[ M\left(\frac{4ac}{z^2}\right) = \frac{16a^2 c^2 + 2b \cdot 4acz + 4ac^2}{z^2} = \frac{4ac}{z^2} M(z) . \]

And then, from applications of Items (1.4) and (1.3A), we have that

\[ \frac{4ac}{z^2} M(z) = M\left(\frac{4ac}{z}\right) = M\left(2a \cdot \frac{2c}{z}\right) = 4a Q\left(\frac{2c}{z}\right) , \]
from which we conclude that (for \( z \neq 0 \))

(1.6) \[
Q\left(\frac{2c}{z}\right) = \frac{c}{z^2} M(z).
\]

Items (1.3B), (1.4) and (1.6) are now, hereby, organized together and conveniently displayed as a below boxed bundle of relationships in the spirit of convenience and ease of application.

\[
(1.7) \quad \begin{align*}
A: \quad Q\left(\frac{z}{2a}\right) &= \frac{1}{4a} M(z) \\
B: \quad M\left(\frac{4ac}{z^2}\right) &= \frac{4ac}{z^2} M(z) \\
C: \quad Q\left(\frac{2c}{z}\right) &= \frac{c}{z^2} M(z).
\end{align*}
\]

Appealing to the relationships of (1.7), another interesting functional relationship presents itself. Hence, consider the following

**Theorem 1.** Given a quadratic \( Q \) and its monic adjunct \( M \), suppose that for \( z, w \in \mathbb{C} \): \( zw = 4ac \neq 0 \). Then: \( zM(w) = wM(z) \).

**Proof:** Referencing Item (1.7B), note that, \( M(w) = M\left(\frac{4ac}{z^2}\right) = \frac{4ac}{z^2} M(z) = \frac{w}{z} M(z) = \frac{w}{z} M(z) \). Therefore, it now follows that,

(1.8) \[
zM(w) = wM(z). \quad \square
\]

**Article 02.: Novel Curiosities of \( Q \) and \( M \) Pairs.** By implementing the functional relationships between a quadratic \( Q \) and its monic adjunct \( M \), novel relationships involving the zeros of such \( Q \) and \( M \) polynomial pairs can be developed. We begin with the following theorem demonstrations.

**Theorem 2.** Given a quadratic \( Q \) and its monic adjunct \( M \), suppose that for \( z, w \in \mathbb{C} \): \( zw = 4ac \neq 0 \). Then: \( M(z) = 0 \iff M(w) = 0 \).

**Proof:** Given the hypothesis, (1.8) is applicable. Also, \( z \neq 0 \neq w \). Hence, by (1.8): \( wM(z) = 0 \iff zM(w) = 0 \). Thus, we have that, \( M(z) = 0 \iff wM(z) = 0 \iff zM(w) = 0 \iff M(w) = 0 \). Therefore: \( M(z) = 0 \iff M(w) = 0 \). \( \square \)

**COROLLARY 2.1** If \( zw - 4ac = 0 \) for two non-zero values, \( z, w \in \mathbb{C} \), then: Either both are zeros of \( M \), or else, Neither is a zero of \( M \).

Zeros of a quadratic polynomial, \( Q \), and those of its monic adjunct, \( M \), can now be related to each other by applying--in concert with each other--the results of Item (1.7A) together with Theorem 2.
Theorem 3. Given a quadratic $Q$ and its monic adjunct $M$, suppose that $M(z) = 0$ for $z \in \mathbb{C}$ where $z \neq 0$. Then: $Q\left(\frac{z}{2a}\right) = 0 = Q\left(\frac{c}{z}\right)$.

Proof: Given that $M(z) = 0$ and $z \neq 0$, and then appealing to Items (1.7A) and (1.7C), it follows that: $Q\left(\frac{z}{2a}\right) = \frac{1}{4a} M(z) = 0 = \frac{c}{z} M(z) = Q\left(\frac{c}{z}\right)$. □

COROLLARY. Given a quadratic (polynomial) $Q(x)$, suppose that $z \neq 0$, and that $z$ is a fixed zero of the monic adjunct $M(x)$. Then, zeros of $Q(x)$ are given by the formulas

$$
(2.1) \quad x_1 = \frac{z}{2a} \quad \text{and} \quad x_2 = \frac{2c}{z}, \quad (z \neq 0).
$$

Theorem 4. Given a quadratic $Q$ and its monic adjunct $M$, suppose that $M(z) = 0$ for $z \in \mathbb{C}$ where $z \neq 0$. Then, the zeros of $Q(x)$ given by the formulas in Item (2.1) are distinct zeros iff $z \neq -b$.

Proof: We proceed to demonstrate that: $x_1 = x_2 \iff z = -b$.

Hence, for $z \neq 0$ and $M(z) = 0$, suppose that $x_1 = x_2$. Then,

$$
x_1 = x_2 \Rightarrow \frac{z}{2a} = \frac{2c}{z} \Rightarrow z^2 - 4ac = 0.
$$

Also, since $M(z) = 0$,

$$
z^2 + 2bz + 4ac = 0.
$$

Adding equations in $z$, we arrive at

$$
0 = 2z^2 + 2bz = 2z(z + b).
$$

Given $z \neq 0$, it we conclude $z = -b$.

Consequently, the implication: $x_1 = x_2 \Rightarrow z = -b$ follows.

Conversely, now suppose that $z = -b$. Then, we see that

$$
0 = M(z) = M(-b) = (-b)^2 + 2b(-b) + 4ac = -b^2 + 4ac.
$$

Thus, $z = -b \Rightarrow -b^2 + 4ac = 0 \Rightarrow -b = \frac{2c}{b} \Rightarrow x_1 = x_2$.

Therefore, we have established: $x_1 = x_2 \iff z = -b$. Hence, the inverse biconditional: $x_1 \neq x_2 \iff z \neq -b$ can be concluded. □

Theorem 5. Given a quadratic (polynomial) $Q(x)$, suppose that $z \neq 0$, and that $z$ is a fixed zero of the monic adjunct $M(x)$. Then, zeros of $Q(x)$, $x_1$ and $x_2$, given by the formulas of Item (2.1) satisfy the following equalities:

$$
(2.2) \quad (i) \quad x_1 + x_2 = -\frac{b}{a} \quad \text{and} \quad (ii) \quad x_1 \cdot x_2 = \frac{c}{a}.
$$

Proof: Given that $M(z) = 0$ and $z \neq 0$, and observing that

$$
0 = M(z) = z^2 + 2bz + 4ac \Rightarrow z^2 + 4ac = -2bz.
$$

Hence, $x_1 + x_2 = \frac{z}{2a} + \frac{2c}{z} = \frac{z^2 + 4ac}{2az} = \frac{-2bz}{2az} = -\frac{b}{a}$.

And, also, $x_1 \cdot x_2 = \frac{z}{2a} \cdot \frac{2c}{z} = \frac{c}{a}$. □
In order to present the developments (and observations) of the preceding Articles as a conveniently organized list of itemized results, a summary of such results is now displayed below.

**Summary of Established Results and Observations:**

(2.3) For \( z \in \mathbb{C} \) and \( z \neq 0 \): \( z \) is a zero of \( \frac{z}{2a} \) iff \( \frac{z}{2a} \) is a zero of \( Q \).

(2.4) For \( z \in \mathbb{C} \) and \( z \neq 0 \): \( z \) is a zero of \( M \) iff \( \frac{4ac}{z} \) is a zero of \( M \).

(2.5) For \( z \in \mathbb{C} \) and \( z \neq 0 \), suppose that \( z \) is a zero of \( M \), then:

\[
(A): \frac{z}{2a} \quad \text{and} \quad \frac{2c}{z} \quad \text{are zeros of} \quad Q \quad \| \quad (B): \text{Distinct zeros} \iff z \neq -b.
\]

(2.6) For \( z \in \mathbb{C} \) and \( z \neq 0 \), suppose that \( z \) is a zero of \( M \); then, the zeros of \( Q \), \( x_1 \) and \( x_2 \), displayed in Item (2.1) satisfy the equalities

\[
(i) \quad x_1 + x_2 = -\frac{b}{a} \quad \text{and} \quad (ii) \quad x_1 \cdot x_2 = \frac{c}{a}.
\]

(2.7) Note that the condition, \( 4ac \neq 0 \), restricts \( Q \) from having zero values for either the leading coefficient, \( a \), or the constant term, \( c \). Of course, the condition that \( a \neq 0 \) is implicit; otherwise, \( Q \) would not be a quadratic. Observe that \( c \neq 0 \) excludes the instance of with a which has a zero value.

(2.8) The condition that \( b^2 + c^2 \neq 0 \) would restrict the admittance of the trivial \( Q \), \( ax^2 = 0 \). Hence, exclusion of such trivial \( Q \) can be assured by considering only admissible quadratic (polynomials) \( Q \) as defined by the following

(2.9) **DEFINITION:** Degenerate and trivial quadratics, \( Q \), can be excluded by considering only admissible quadratics, \( Q \), which satisfy the condition that: \( a \cdot (b^2 + c^2) \neq 0 \).

**Article 03.: Alternative Renditions to the Quadratic Formula.** By appealing to the results of the preceding Articles, novel and alternative renditions to the standard quadratic formula can be developed and formulated. In order to accomplish this objective, we introduce the following expression references.

**DEFINITION:** Given a general quadratic equation, \( Q(x) = 0 \), the expressions created by the plus and minus sign options in the formula,

\[
E = b \pm \sqrt{b^2 - 4ac}
\]

are hereby defined as quadratic root Effectors.
We now note the following observations regarding quadratic root effectors:

(i) Degenerate and trivial quadratics are avoided by considering only admissible forms which satisfy the condition: \( a \cdot (b^2 + c^2) \neq 0 \) (as detailed in Item (2.9)).

(ii) \( E \neq 0 \), for at least one of the root effector options (for admissible forms).

**Theorem 6.** The roots of an admissible quadratic equation, \( Q(x) = 0 \), can be determined by using the following quadratic root effector formulas (QREF) with a non-zero root effector.

\[
x_1 = -\frac{E}{2a} \quad \text{and} \quad x_2 = -\frac{2c}{E}.
\]

**Proof:** From (3.2) we see that there does exist at least one non-zero root effector option of that admissible \( Q \). Further, we note that for the monic adjunct, \( M \), we have that \( M(x) = x^2 + 2bx + b^2 - (b^2 - 4ac) = (x + b)^2 - (b^2 - 4ac) \), and, consequently, \( M(-E) = [(-b \pm \sqrt{b^2 - 4ac}) + b]^2 - (b^2 - 4ac) = 0 \), for either option of the root effectors. Now, by applying an \( E \neq 0 \) option, \( (-E) \neq 0 \) and \( M(-E) = 0 \), together with (2.1) of Theorem 3, establishes Item (3.3), and so, completes this proof.

**Theorem 7.** The roots of an admissible quadratic equation, \( Q(x) = 0 \), which are formulated by the root effector formulas of Item (3.3) satisfy the equalities

\[
\begin{align*}
\text{(i)} \quad x_1 + x_2 &= -\frac{b}{a} \quad \text{and} \quad \text{(ii)} \quad x_1 \cdot x_2 &= \frac{c}{a}.
\end{align*}
\]

**Proof:** Here, we simply apply Theorem 5 where: \( z = (-E) \neq 0 \) and observing that \( M(-E) = 0 \) as developed in the proof of Theorem 6.

Having observed the fundamental relationships and connective interplay between a quadratic \( Q \) and its monic adjunct \( M \) as presented in the preceding Articles, this next result should actually come as no surprise.

**Theorem 8.** The roots of an admissible quadratic equation, \( Q(x) = 0 \), as determined by quadratic root effector formulas of Item (3.3), can be classified by the monic adjunct discriminant \( M(-b) \) as follows:

\[
\begin{align*}
\text{(i)} \quad M(-b) < 0 & \Rightarrow \text{distinct, real roots} \\
\text{(ii)} \quad M(-b) = 0 & \Rightarrow \text{one, real root} \\
\text{(iii)} \quad M(-b) > 0 & \Rightarrow \text{complex conjugate roots}
\end{align*}
\]

**Proof:** Note that \( M(-b) = (-b)^2 + 2b(-b) + 4ac = -b^2 + 4ac \). Now, by referencing the root effectors formula in Item (3.1) and observing that since \( -M(-b) \) is, in fact, the radicand thereof, then the parity and value of \( M(-b) \) clearly classify the roots of \( Q(x) = 0 \) as described in Item (3.5).
EXAMPLE #01 REGARDING QUADRATIC ADJUNCTS

(* Now consider the Given Quadratic:  \( Q(x) = 3x^2 - 5x + 2 \). Then, 
\( M(x) = x^2 - 10x + 24 \). Now proceed with the solution process.
The symbol "E" is Protected. Hence, we use "R" to designate E *)

Clear[a, b, c, R, x1, x2]
Q[x_] := a x^2 + b x + c
M[x_] := x^2 - 10 x + 24
a = 3; b = -5; c = 2;
SequenceForm["M[-b] = ", M[-b], ", < 0 "]
SequenceForm["R \in ", \{ b + \sqrt{b^2 - 4 a c} , b - \sqrt{b^2 - 4 a c} \} ]
M[-b] = -1 < 0
R \in \{-4, -6\}

M[-b] = -1 < 0  (* Monic Discriminant < 0 ==> Two Real Zeros *)

(* Select R = -4, and then calculate x1 and x2 *)
R = -4;
SequenceForm["x1 = ", \(-R\) \[ \frac{2}{a} \], ", and ", "x2 = ", \[ \frac{2 c}{\(-R\)} \] ]

x1 = \[ \frac{2}{3} \] and x2 = 1

(* Checking these x-values, ... we find that *)
SequenceForm["Q(x1) = ", Q[\[ \frac{2}{3} \]], ", and ", "Q(x2) = ", Q[1] ]
Q(x1) = 0 and Q(x2) = 0

Q(x1) = 0 and Q(x2) = 0  (* Illustration #01 is COMPLETE *)

(*==================================================================*)
EXAMPLE #02 REGARDING QUADRATIC ADJUNCTS

(* Now suppose that: \( Q(x) = 9x^2 - 12x + 5 \). Then, \( M(x) = x^2 - 24x + 180 \).
Proceeding with symbol "R" to designate \( E \), we have that ... *)

Clear[a, b, c, R, x1, x2]
Q[x_] := a x^2 + b x + c
M[x_] := x^2 + 2 b x + 4 a c
a = 9; b = -12; c = 5;
SequenceForm["M(-b) = " , M[-b], " > 0 "]
SequenceForm["R \in " , \{ b + \sqrt{b^2 - 4 a c} , b - \sqrt{b^2 - 4 a c} \} ]

\[ M(-b) = 36 > 0 \]

\[ M(-b) = 36 > 0 \ (\text{\large Monic Discriminant} > 0 \Rightarrow \text{Complex Zeros}) \]

\[ R \in \{-12 + 6 \text{I}, -12 - 6 \text{I}\} \]

(* Select \( R = -12 + 6 \text{I} \), and then calculate \( x1 \) and \( x2 \) *)
R = -12 + 6 I;

SequenceForm["x1 = " , \(-\text{R}\) \(\frac{2 c}{ \text{(-R)}}\) , " and " , "x2 = " , \(\frac{2 c}{ \text{(-R)}}\) ]

\[ x1 = \frac{2}{3} - \frac{I}{3} \quad \text{and} \quad x2 = \frac{2}{3} + \frac{I}{3} \]

(* Checking these \( x \)-values, ... we find that *)

SequenceForm["Q(x1) = ", Q[\(\frac{2}{3} - \frac{I}{3}\)] , " and " , "Q(x2) = ", Q[\(\frac{2}{3} + \frac{I}{3}\)] ]

\[ Q(x1) = 0 \quad \text{and} \quad Q(x2) = 0 \]

\[ Q(x1) = 0 \quad \text{and} \quad Q(x2) = 0 \ (\text{\large Illustration \#01 is COMPLETE}) \]
EXAMPLE #03 REGARDING QUADRATIC ADJUNCTS

(* Imagine a Quadratic Polynomial Q(x) over the Finite Field of integers Modulo (13), GF (13), and denoted by the symbol Z_{13}. This illustration computes the ZEROS of the Given Quadratic, Q(x) = 12x^2+5x+9, over Z_{13}, by implementing the methods of this composition.

===================================================================== 

In the spirit of convenience, ... both: (SqrList)-- the Squares of Z_{13} and, also, (InvList)-- the Multiplicative Inverses of Z_{13}, are hereby listed in order of: SqrList = {1^2 through to 13^2}; AND InvList = {1^{-1} through to 13^{-1}} , respectively. Now, ... consider the following results from Mathematica. 

Note: The Symbol "E" is PROTECTED in Mathematica; here, the Symbol "R" will INSTEAD be used AS A SUBSTITUTE for the SYMBOL "E " in what follows here. *)

SqrList = PowerMod[{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}, 2, 13]; 
InvList = PowerMod[{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}, -1, 13]; 
SequenceForm["SqrList = ", SqrList] 
SequenceForm[ "InvList = ", InvList] 
(* And, so ... *)

SqrList = {1, 4, 9, 3, 12, 10, 10, 12, 3, 9, 4, 1}

InvList = {1, 7, 9, 10, 8, 11, 2, 5, 3, 4, 6, 12}

DETAILS & DEVELOPMENT CONTINUED ON NEXT PAGE
Applying the Results of this Composition to Q(x), below, note that

\[ Q[x_] := 12 x^2 + 5 x + 9 \]
Clear[a, b, c, R, x1, x2]
a = 12; b = 5; c = 9;
SequenceForm["R \in \{ a \}, \{ b + \sqrt{Mod[b^2 - 4 a c, 13]}, b - \sqrt{Mod[b^2 - 4 a c, 13]} \}""]

Hence, we observe that
R \in \{8, 2\}

Therefore, applying the Root Effector Formulas by selecting R = 8,
R = 8;
x1 = Mod\left[\frac{(-R)}{2a}, 13\right]; x2 = Mod\left[\frac{2c}{(-R)}, 13\right];
SequenceForm["x1 = ", x1, ", " , "x2 = ", x2]

And so, we have that ...

\[ x1 = \frac{38}{3} \text{ and } x2 = \frac{43}{4} \]

So that \( Mod(13), x1 = 38*3^{-1} \text{ and } x2 = 43*4^{-1} \)

Consequently it follows that .................................................

\[ x1 = 4 \text{ and } x2 = 1 \]

These x-values ARE ZEROS via the above Mathematica Code ...............

Hence, THIS EXAMPLE #03 IS COMPLETE ........................................