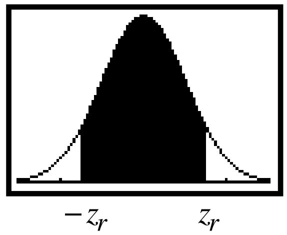
MATH 109

z–scores and Confidence Intervals

Given a "Standard Bell Curve Z" having mean 0 and standard deviation 1, a z–score is the value z_r such that $P(-z_r \le Z \le z_r) = r$. That is, there is probability r between the points $-z_r$ and $+z_r$.



Prob = r	**z-score = z_r**
0.90	1.645
0.95	1.96
0.98	2.326
0.99	2.576

These commonly used z–scores should be memorized.

Confidence Interval for the Mean

The z–scores are used to find confidence intervals for the true unknown mean μ of a population. To estimate μ, we first conduct a random sample of size n from the population, and then compute the sample mean $\bar{x}$ and sample deviation S.

We would like to say that μ is about $\bar{x}$ ± some margin of error. We can never be 100% sure if the unknown μ will really be within our margin of error. But with larger sample sizes, we have a higher probability that μ will be within our bounds.

For *large* samples, the confidence interval for μ, having level of confidence r, is given by

$$\mu \approx \bar{x} \pm \frac{z_r \times S}{\sqrt{n}}$$

or in interval form: $\bar{x} - \frac{z_r \times S}{\sqrt{n}} \le \mu \le \bar{x} + \frac{z_r \times S}{\sqrt{n}}$.

The z–score z_r is chosen depending on the desired level of confidence r. For instance, by using $z_r = 1.96$, we would have a "95% confidence interval." Then if we were to obtain random samples of size n over and over, we would be obtaining different $\bar{x}$ and S each time thereby resulting in different intervals. But about 95% of the time, the true mean μ should be within our interval.

Example 1. A random sample of 100 ACT scores of students applying to Western yields $\overline{x} = 21.2$ with a sample deviation of 4.46. Find a 90% confidence interval for the true of ACT score μ of all applicants.

Solution. Here the appropriate z–score is 1.645; thus, $\mu \approx \overline{x} \pm \frac{z_r \times S}{\sqrt{n}} = 21.2 \pm \frac{1.645 \times 4.46}{\sqrt{100}} \approx 21.2 \pm 0.734$. Therefore we can say with "90% confidence" that $20.466 \leq \mu \leq 21.934$.

Exercises

1. A study was conducted of recent high school graduates who began full-time jobs rather than going to college. We wish to find the average starting income μ of such workers. A random sample of size 60 gave a sample mean of $24,500 with a sample deviation of $2350. Find a 95% confidence interval for μ.

2. A study was conducted on the IQ levels of musically precocious children. The sample mean score of 30 such children was 132 with a sample deviation of 9.45. Find a 98% confidence interval for the average IQ score of all such musically precocious children.

3. A survey of 200 entering Freshmen at WKU found that the average number of credit hours enrolled was 16.58 with a sample deviation of 2.46. Find a 99% confidence interval for the average number of hours enrolled for all Freshmen.

Confidence Interval for a Population Proportion p

The z–scores also are used to find confidence intervals for a true unknown population proportion p regarding a "Yes/No" question. To estimate p, we first conduct a random sample of size n from the population, and then compute the sample proportion given by $\overline{p} = \frac{\#\ \text{“Yes”}}{\#\ \text{Responses}}$.

Again we would like to say that p is about $\overline{p}$ $\pm$ some margin of error. We still can never be 100% sure if the unknown p will really be within our margin of error. But with larger sample sizes, we have a higher probability that p will be within our bounds.

For large samples, the *confidence interval* for p, having level of confidence r, is given by

$$p \approx \overline{p} \pm \frac{z_r \times 0.5}{\sqrt{n}}$$

or in interval form: $\overline{p} - \frac{z_r \times 0.5}{\sqrt{n}} \leq p \leq \overline{p} + \frac{z_r \times 0.5}{\sqrt{n}}$.

Example 2. In a random sample of 900 adults, 513 approved of the "war on terror." Find a 95% confidence interval for the true proportion of adults who approve.

Solution. Here the appropriate z–score is 1.96. The sample proportion is $\overline{p} = 513/900 = 0.57$. The 95% confidence interval is then $p \approx 0.57 \pm \dfrac{1.96 \times 0.5}{\sqrt{900}} \approx 0.57 \pm 0.0327$. So with "95% confidence" we have $0.5373 \le p \le 0.6027$, which means that from 53.73% to 60.27% of adults approve of the war on terror.

Often you see news reports that state something like "57% of adults favor the war on terror." Then there is a small disclaimer that states "this result was based on a random sample of 900 adults and has a margin of error of ± 3.237 percentage points."

Exercises

4. (a) In a statewide poll, 103 out of 500 Kentuckians rated themselves as die-hard UK basketball fans. Find a 90% confidence interval for the true proportion of Kentuckians who were die-hard UK fans at that time.

(b) At the same time, a survey of Western's students found that 84 out of 300 students rated themselves as die-hard UK basketball fans. Find a 99% confidence interval for the true proportion of students who were die-hard UK fans at that time.

5. A poll commissioned by the Center on Addiction and Substance Abuse at Columbia University found that 1340 out of 2000 adults interviewed believed that popular culture encourages drug use. Find a 98% confidence interval for the true proportion of adults nationwide having this belief.

Solutions

1. $\mu \approx 24{,}500 \pm \dfrac{1.96\times 2350}{\sqrt{60}} = \$24{,}500 \pm \$594.63$; or $\$23{,}905.37 \le \mu \le \$25{,}094.63$.

2. $\mu \approx 132 \pm \dfrac{2.326\times 9.45}{\sqrt{30}} = 132 \pm 4.013 \approx 132 \pm 4$; or $128 \le \mu \le 136$.

3. $\mu \approx 16.58 \pm \dfrac{2.576\times 2.46}{\sqrt{200}} = 16.58 \pm 0.448$; or 16.132 hrs $\le \mu \le$ 17.028 hrs.

4. (a) The sample proportion is $\overline{p} = 103/500 = 0.206$ and the z–score for a 90% confidence interval is $z_r = 1.645$. Thus, the true proportion of Kentuckians who were die-hard UK fans at the time was

$$p \approx 0.206 \pm \frac{1.645\times 0.5}{\sqrt{500}} \approx 0.206 \pm 0.0368,$$

or $0.1692 \le p \le 0.2428$. So from about 16.92% to 24.28% of Kentuckians were die-hard UK fans.

(b) Now the sample proportion is $\overline{p} = 84/300 = 0.28$. Using the 0.99 z–score of 2.576, we now have

$$p = 0.28 \pm \frac{2.576\times 0.5}{\sqrt{300}} \approx 0.28 \pm 0.07436.$$

So at that time, the true percentage of Western students who were die-hard UK fans was about from 20.564% to 35.436%.

5. The sample proportion is $\overline{p} = 1340/2000 = 0.67$. The z–score for a 98% confidence interval is $z_r = 2.326$. Thus, the confidence interval for the true proportion p is

$$p \approx 0.67 \pm \frac{2.326\times 0.5}{\sqrt{2000}} \approx 0.67 \pm 0.026,$$

or $0.644 \le p \le 0.696$.

So somewhere from 64.4% to 69.6% of adults think popular culture encourages drug use.