Let $A$ and $B$ be non-empty sets. Here are the basic definitions, concepts, and theorems we need as background to study functions:

**Definition 4.1.** The cross-product $A \times B$ is the set of all ordered pairs from $A$ and $B$:

\[ A \times B = \{(a, b) : a \in A, b \in B\}. \]

In particular, the $xy$ plane is $R^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}\}$. The integer lattice is the subset of $R^2$ given by $Z \times Z = \{(m, n) : m \in Z, n \in Z\}$.

**Definition 4.2.** A relation $R$ from $A$ to $B$ is any subset of $A \times B$. If $(a, b) \in R$, then we write $a R b$, which means “$a$ is related to $b$.”

For example, in $R^2$ let $R = \{(x, y) : x^2 + y^2 = 1\}$ (i.e., the unit circle). Then $(0, 1) \in R$ and $(0, -1) \in R$; hence, 0 is related to 1 and 0 is related to $-1$.

**Definition 4.3.** Let $D$ be a non-empty subset of $A$. A function $f$ on $A \times B$ is a relation $R$ from $D$ to $B$ such that

(i) For every $d \in D$, there exists an element $b \in B$ such that $d R b$, and

(ii) If $d R b$ and $d R c$, then $b = c$.

The set $D$ is called the domain of the function. In the context of a function, set $B$ is called the co-domain. In this case, we write $f : D \to B$ and write $f(d) = b$ whenever $d R b$. The above conditions then become

(i) For every $d \in D$, there exists an element $b \in B$ such that $f(d) = b$, and

(ii) If $f(d) = b$ and $f(d) = c$, then $b = c$.

Condition (i) means that the function must be defined for all elements in the domain. Condition (ii) means that it must be well-defined; that is, an element $d$ in the domain can be assigned to only one element in the co-domain.

For example, on $R^2$ consider the function $f(x) = \sqrt{x - 4}$. Because the function is only defined for $x \geq 4$, the domain is $D = [4, \infty)$. So the proper way to define the function is to write $f : [4, \infty) \to \mathbb{R}$ given by $f(x) = \sqrt{x - 4}$. Here, $\mathbb{R}$ is the co-domain, but not necessarily the range.

The unit circle can be written as $y = \pm \sqrt{1 - x^2}$ where $D = [-1, 1]$. But this relation is not a function because every $x \in (-1, 1)$ is assigned to two values in the co-domain. (For example, when $x = 1/2$, then $y = \pm \sqrt{3/2}$.)
Definition 4.4. Let \( f : D \rightarrow B \) be a function on \( A \times B \). The range of \( f \) is the set of elements in \( B \) for which there exists an element \( d \in D \) such that \( f(d) = b \):

\[
\text{Range } f = \{ b \in B : f(d) = b \text{ for some } d \in D \}.
\]

If \( f(d) = b \), then \( d \) is called a pre-image of \( b \) under \( f \).

Definition 4.5. Let \( f : D \rightarrow B \) be a function on \( A \times B \). Then \( f \) is onto (or surjective) if \( \text{Range } f = B \). That is, for every \( b \in B \) there is at least one pre-image \( d \in D \) such that \( f(d) = b \).

For example, consider the functions

\[
f: \mathbb{R} \rightarrow \mathbb{R} \text{ given by } f(x) = e^x \quad \text{and} \quad g : (0, \infty) \rightarrow \mathbb{R} \text{ given by } g(x) = \ln x.
\]

Then \( \text{Range } f = (0, \infty) \) because \( e^x > 0 \) for all \( x \). Hence, \( f \) is not onto all of \( \mathbb{R} \). But \( \text{Range } g = (-\infty, \infty) = \mathbb{R} \); hence, \( g \) is onto.

Likewise, for the functions \( f(x) = x^2 \) and \( g(x) = x^3 \) on \( \mathbb{R}^2 \), then \( g \) is onto having range \((-\infty, \infty)\), but \( f \) is not onto having range \([0, \infty) \neq \mathbb{R}\).

Definition 4.6. Let \( f : D \rightarrow B \) be a function on \( A \times B \). Then \( f \) is one-to-one (1–1), also called injective, provided the following condition holds:

For all \( d_1, d_2 \in D \), if \( d_1 \neq d_2 \), then \( f(d_1) \neq f(d_2) \).

By the contrapositive, it is equivalent to say that \( f \) is one-to-one provided:

For all \( d_1, d_2 \in D \), if \( f(d_1) = f(d_2) \), then \( d_1 = d_2 \).

To prove that a function is 1–1, we often use the second form of Definition 4.6. For example, consider \( f: \mathbb{R} \rightarrow \mathbb{R} \) given by \( f(x) = 2x - 8 \). We prove that \( f \) is 1–1 as follows:

Assume \( f(x_1) = f(x_2) \). Then \( 2x_1 - 8 = 2x_2 - 8 \), which implies \( 2x_1 = 2x_2 \) and then \( x_1 = x_2 \). Hence, \( f \) is 1–1.

To show that a function is not 1–1, it suffices to find one counterexample where \( d_1 \neq d_2 \) but \( f(d_1) = f(d_2) \). For example, \( f(x) = x^2 \) is not 1–1 because \( 5 \neq -5 \) but \( f(5) = 25 = f(-5) \). (Here, \( x^2 \) is two-to-one: Two \( x \)-values are mapped to the same \( y \).)

Lemma 4.1. Let \( D \subseteq \mathbb{R} \) and let \( f: D \rightarrow \mathbb{R} \) be a function. If \( f \) is strictly increasing or strictly decreasing, then \( f \) is one-to-one.

Proof. Assume \( f \) is strictly increasing, and suppose \( x_1, x_2 \in D \) with \( x_1 \neq x_2 \). Without loss of generality, we may assume \( x_1 < x_2 \). Because \( f \) is strictly increasing, we must have \( f(x_1) < f(x_2) \). That is, \( x_1 \neq x_2 \) implies \( f(x_1) \neq f(x_2) \); thus \( f \) is one-to-one. The argument is similar if \( f \) is strictly decreasing.
**Definition 4.7.** Let \( f: D \to B \) be a function on \( A \times B \). Then \( f \) is bijective provided \( f \) is both one-to-one and onto. In this case, \( f \) is also called a bijection or a one-to-one correspondence between \( D \) and \( B \).

**Note:** If \( f \) is only one-to-one, then we may consider the function as \( f: D \to \text{Range } f \). Then \( f \) becomes a one-to-one correspondence between the domain and \( \text{Range } f \).

For example, consider \( f: \mathbb{R} \to \mathbb{R} \) given by \( f(x) = e^x \). Then \( f \) is strictly increasing, so \( f \) is 1–1. But \( f \) is not onto because the range is \((0, \infty)\). But then we can re-write \( f \) as \( f: \mathbb{R} \to (0, \infty) \) defined by \( f(x) = e^x \). Now \( f \) is both 1–1 and onto, so we have a 1–1 correspondence between the entire real line \( \mathbb{R} \) and the half line \((0, \infty)\).

Every real number \( x \) in \((-\infty, \infty)\) is mapped to one and only one number \( y \) in \((0, \infty)\). And different \( x \) must be mapped to different \( y \). But how is the “bigger” set \((-\infty, \infty)\) collapsed into the “smaller” set \((0, \infty)\) without some different \( x \) going to the same \( y \)?

First, 0 is mapped to 1. Then all of \((-\infty, 0)\) is mapped onto the interval \((0, 1)\) on the \( y \)-axis. And all of \((0, \infty)\) is mapped onto \((1, \infty)\) on the \( y \)-axis.

The one-to-one correspondence between \((-\infty, \infty)\) and \((0, \infty)\) means that these sets are really equivalent in terms of cardinality. But for finite sets \( A, B \), it is impossible to collapse a larger set into a smaller set in a 1–1, onto fashion.

**Definition 4.8.** Let \( f: D \to \text{Range } f \) be a bijection. Then \( f \) has an inverse, denoted \( f^{-1} \), defined as follows: \( f^{-1}: \text{Range } f \to D \) is computed by \( f^{-1}(b) = d \) where \( f(d) = b \).

**Claim 1:** \( f^{-1}: \text{Range } f \to D \) is well-defined.

**Proof.** Suppose \( f^{-1}(b) = d_1 \) and \( f^{-1}(b) = d_2 \). We must show that \( d_1 = d_2 \) (i.e., that \( f^{-1} \) can map an element to only one value.) By the definition of \( f^{-1} \), if \( f^{-1}(b) = d_1 \) and \( f^{-1}(b) = d_2 \), then \( f(d_1) = b \) and \( f(d_2) = b \). But because \( f \) is 1–1, we have \( d_1 = d_2 \).

**Claim 2:** \( f^{-1}: \text{Range } f \to D \) is one-to-one.
Proof. Suppose \( f^{-1}(b_1) = d = f^{-1}(b_2) \). By definition of \( f^{-1} \), \( f(d) = b_1 \) and \( f(d) = b_2 \). But because \( f \) is a (well-defined) function, it must be the case that \( b_1 = b_2 \). Hence, \( f^{-1} \) is one-to-one.

**Claim 3:** \( f^{-1} : \text{Range } f \to D \) is onto.

Proof. Let \( d \in D \). Then let \( b = f(d) \in \text{Range } f \). By definition of \( f^{-1} \), \( f^{-1}(b) = d \). Hence every \( d \in D \) is in \( \text{Range } f^{-1} \); thus, \( f^{-1} \) is onto.

(Note now that \( \text{Domain } f^{-1} = \text{Range } f \) and \( \text{Range } f^{-1} = D = \text{Domain } f \).)

**Claim 4:** For all \( d \in D \), \( f^{-1}(f(d)) = d \).

Proof. Let \( d \in D \) and let \( b = f(d) \). Then \( f^{-1}(f(d)) = f^{-1}(b) = d \) (by definition of \( f^{-1} \)).

**Claim 5:** For all \( b \in \text{Range } f \), \( f(f^{-1}(b)) = b \).

Proof. Let \( b \in \text{Range } f \). Then there exists \( d \in D \) such that \( f(d) = b \). By definition of \( f^{-1} \), we have \( f^{-1}(b) = d \). Then \( f(f^{-1}(b)) = f(d) = b \).

**Example 4.1.** Let \( A = N_5 = \{1, 2, 3, 4, 5\} \) and \( B = \{a, b, c, d, e\} \). Define \( f : N_5 \to B \) by

\[
\begin{align*}
  f(1) &= c & f(2) &= e & f(3) &= a & f(4) &= b & f(5) &= d
\end{align*}
\]

Then \( f \) is both 1–1 and onto. So \( f^{-1} : B \to N_5 \) is also a bijection and is defined by

\[
\begin{align*}
  f^{-1}(a) &= 3 & f^{-1}(b) &= 4 & f^{-1}(c) &= 1 & f^{-1}(d) &= 5 & f^{-1}(e) &= 2
\end{align*}
\]

**Example 4.2.** Define \( f : \mathbb{N} \to \mathbb{N} \) by \( f(n) = 2n \). Then \( f \) is 1–1 (if \( f(n_1) = f(n_2) \) then \( n_1 \) must equal \( n_2 \)). However \( f \) is not onto (no odd numbers are in the range). If we let \( E^+ \) denote the positive even numbers, then we can redefine the function as \( f : \mathbb{N} \to E^+ \) given by \( f(n) = 2n \). Now \( f \) is both 1–1 and onto. And \( f^{-1} : E^+ \to \mathbb{N} \) is defined by \( f^{-1}(k) = k / 2 \). So the natural numbers \( \{1, 2, 3, 4, \ldots\} \) are in one-to-one correspondence with the positive even numbers \( \{2, 4, 6, 8, \ldots\} \).

\[
\begin{array}{cccccccc}
  N & 1 & 2 & 3 & 4 & \ldots & n & \ldots \\
  \downarrow & \uparrow & \downarrow & \uparrow & \downarrow & \uparrow & \ldots & \downarrow & \uparrow & \ldots \\
  E^+ & 2 & 4 & 6 & 8 & \ldots & 2n & \ldots
\end{array}
\]

Even though \( E^+ \) is a proper subset of \( \mathbb{N} \) and \( \{2, 4, 6, 8, \ldots\} \) seems to be a “smaller” set than \( \{1, 2, 3, 4, 5, 6, 7, 8, \ldots\} \) having half the size, in fact, \( E^+ \) and \( \mathbb{N} \) are the same size and have the same number of elements. That is, they have the same cardinality.
Example 4.3. Define $f: (-\pi/2, \pi/2) \to \mathbb{R}$ by $f(x) = \tan x$. Then $f$ is strictly increasing; hence, $f$ is 1–1. The range of $f$ is $(-\infty, \infty) = \mathbb{R}$; hence, $f$ is onto. Thus, $f^{-1}$ exists and $f^{-1}: \mathbb{R} \to (-\pi/2, \pi/2)$ is the “arctangent” function: $f^{-1}(x) = \tan^{-1}(x) = \arctan(x)$. For instance, $\tan(\pi/4) = 1$; hence, $\arctan(1) = \pi/4$.

Here we see that an interval of finite length $(-\pi/2, \pi/2)$ is in one-to-one correspondence with an interval of infinite length $(-\infty, \infty)$. So these intervals really have the same “size” in terms of cardinality or number of elements.

Example 4.4. We assert that any open interval $(a, b)$ is in one-to-one correspondence with any other open interval $(c, d)$.

To show this result, we will define a linear function $f: (a, b) \to (c, d)$. The slope will be $m = \frac{d - c}{b - a}$. Then using point/slope, we have $y - c = \left(\frac{d - c}{b - a}\right)(x - a)$; thus we obtain

$$f: (a, b) \to (c, d) \text{ defined by } f(x) = \left(\frac{d - c}{b - a}\right)(x - a) + c$$

Note: We can use the same function to show that the closed interval $[a, b]$ is in one-to-one correspondence with any closed interval $[c, d]$.

Example 4.5. We have noted before that $f: \mathbb{R} \to (0, \infty)$ defined by $f(x) = e^x$ is a bijection. Then $f^{-1}: (0, \infty) \to \mathbb{R}$ given by $f^{-1}(x) = \ln x$ is also a bijection. Note the property of the compositions:

For all $x \in \mathbb{R}$, $f^{-1}(f(x)) = \ln e^x = x$

For all $x > 0$, $f(f^{-1}(x)) = e^{\ln x} = x$

Again we note that $(-\infty, \infty)$ is in one-to-one correspondence with $(0, \infty)$ by means of either $f: \mathbb{R} \to (0, \infty)$ given by $f(x) = e^x$ or $g:(0, \infty) \to \mathbb{R}$ given by $g(x) = \ln x$.

Henceforth we will assume that the domain of any function is the entire set $A$ and simply write $f: A \to B$. If $f$ is a bijection, then the inverse is denoted by $f^{-1}: B \to A$. 
Definition 4.9. Let \( f : A \rightarrow B \) and \( g : B \rightarrow C \) be functions. We define the composition \( g \circ f : A \rightarrow C \) by \((g \circ f)(a) = g(f(a))\).

Theorem 4.1. Let \( f : A \rightarrow B \) and \( g : B \rightarrow C \) be functions and let \( g \circ f : A \rightarrow C \) be the composition. If \( f \) and \( g \) are both one-to-one and onto, then \( g \circ f \) is also one-to-one and onto.

Proof. One-to-one: Assume \((g \circ f)(a) = (g \circ f)(b)\); that is, \(g(f(a)) = g(f(b))\). Because \( g \) is one-to-one, we have \(f(a) = f(b)\). Then because \( f \) is one-to-one, we have \(a = b\). Hence, \(g \circ f\) is one-to-one.

Onto: Let \( c \in C \). Because \( g \) is onto, there exists \( b \in B \) such that \( g(b) = c \). Then because \( f \) is onto, there exists \( a \in A \) such that \( f(a) = b \). Then \((g \circ f)(a) = g(f(a)) = g(b) = c\). Hence, \(g \circ f\) is onto.

Now we know that \( g \circ f \) is a bijection whenever \( f \) and \( g \) are both bijections. So \( g \circ f \) has an inverse \((g \circ f)^{-1} : C \rightarrow A\). The following result explains how to compute the inverse of a bijective composition:

Theorem 4.2. Let \( f : A \rightarrow B \) and \( g : B \rightarrow C \) be bijective functions and let \( g \circ f : A \rightarrow C \) be the bijective composition. Then \((g \circ f)^{-1} : C \rightarrow A\) is given by \((g \circ f)^{-1} = f^{-1} \circ g^{-1}\).

Proof. Let \( c \in C \). By the definition of \( g^{-1} \), we have \( g^{-1}(c) = b \) where \( g(b) = c \). Then by the definition of \( f^{-1} \), we have \( f^{-1}(b) = a \) where \( f(a) = b \). Then,

\[
(g \circ f)(a) = g(f(a)) = g(b) = c.
\]

So \((g \circ f)^{-1}(c) = a\) by the definition of \((g \circ f)^{-1}\). But we also have

\[
(f^{-1} \circ g^{-1})(c) = f^{-1}(g^{-1}(c)) = f^{-1}(b) = a.
\]

Thus, \((g \circ f)^{-1}(c) = (f^{-1} \circ g^{-1})(c)\) for all \( c \in C \), which means \((g \circ f)^{-1} = f^{-1} \circ g^{-1}\).

Definition 4.10. Let \( A, B \) be two non-empty sets. We say that \( A \) and \( B \) are equivalent, denoted \( A \sim B \), if there exists a bijection \( f : A \rightarrow B \).

By Examples 1 – 5, we have the following equivalences:

\[
\{1, 2, 3, 4, 5\} \sim \{a, b, c, d, e\} \quad N \sim E^+ \quad (-\pi / 2, \pi / 2) \sim (-\infty, \infty)
\]

\[
(a, b) \sim (c, d) \text{ for any two open intervals} \quad (0, \infty) \sim (-\infty, \infty)
\]

Many more equivalences can be expressed by applying the following theorem:
**Theorem 4.3.** Set equivalence is an equivalence relation.

**Proof.** We must show that \( \sim \) is reflexive, symmetric, and transitive. First, given any non-empty set \( A \), define \( f: A \to A \) by \( f(a) = a \) (the identity function). Then \( f \) is clearly one-to-one and onto; hence, \( f \) is a bijection. So \( A \sim A \) (reflexive).

Next, suppose \( A \sim B \). Then there exists a bijection \( f: A \to B \). But then \( f^{-1}: B \to A \) is also a bijection. Hence, \( B \sim A \) (symmetric).

Finally, suppose \( A \sim B \) and \( B \sim C \). Then there exist bijections \( f: A \to B \) and \( g: B \to C \). But then \( g \circ f: A \to C \) is also a bijection by Theorem 4.1. Hence, \( A \sim C \) (transitive). Thus, \( \sim \) is an equivalence relation.

**Theorem 4.4.** The interval \((0, 1)\) is equivalent to the entire real line \( \mathbb{R} \).

**Proof.** We know that any two bounded, open intervals are equivalent and we have a specific open interval \((-\pi/2, \pi/2)\) that is equivalent to the real line \( \mathbb{R} \) by means of the bijection \( f(x) = \tan x \); thus, \((0, 1) \sim (-\pi/2, \pi/2) \sim (-\infty, \infty)\). By transitivity, \((0, 1) \sim (-\infty, \infty)\).

**Exercises**

1. Explain whether or not each function is one-to-one and whether or not each function is onto. If one is a bijection, find its inverse.

   (a) \( f: \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = |x| \)   
   (b) \( f:[0, \infty) \to [0, 1) \) defined by \( f(x) = \frac{x}{x+1} \)

   (c) \( f: \mathbb{R} \to [-1, 1] \) defined by \( f(x) = \sin x \)   
   (d) \( f:(0, \infty) \to [0, \infty) \) defined by \( f(x) = \frac{1}{x} \)   
   (e) \( f: [3, \infty) \to (-\infty, -2] \) defined by \( f(x) = -4\sqrt{x-3} - 2 \).

2. Define a bijection \( f \) from the closed interval \([2, 6]\) to the closed interval \([10, 20]\). Find its inverse and verify that \( f^{-1}(f(x)) = x \) for \( x \in [2, 6] \), and \( f(f^{-1}(x)) = x \) for \( x \in [10, 20] \).

3. Define a 1-1, onto function \( f:(0, 1) \to \mathbb{R} \).

4. Use set equivalences to prove that \((0, \infty)\) is equivalent to any open interval \((a, b)\).