Let $S = \{ \vec{u}_1, \ldots, \vec{u}_m \}$ be a collection of $m$ distinct vectors in $\mathbb{R}^n$. Each vector has $n$ coordinates. When we write the vectors in column form, then they become $n \times 1$ matrices. Aligning them together into one matrix, we obtain an $n \times m$ matrix representation $A$.

**Example 1.** In $\mathbb{R}^4$, let $\vec{u}_1 = (1, 0, 4, 2)$, $\vec{u}_2 = (3, -2, 0, 6)$, and $\vec{u}_3 = (1, -1, 1, 4)$. Then $A$ is the $4 \times 3$ matrix

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -2 & -1 \\ 4 & 0 & 1 \\ 2 & 6 & 4 \end{bmatrix}.$$ 

This $4 \times 3$ matrix also defines a linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ (which we know cannot be onto because $\mathbb{R}^4$ is larger than $\mathbb{R}^3$, but may or may not be one-to-one.)

### Spanning Sets

**Definition.** Let $S = \{ u_1, \ldots, u_m \}$ be a collection of $m$ distinct vectors in a vector space $V$. Then $S$ is said to span $V$ if every vector in $V$ can be written as a linear combination of the vectors in $S$.

**Recall:** In $\mathbb{R}^n$, in order to determine if a vector $\vec{b}$ is a linear combination of the vectors in $S$, we simply let $B$ be the column form of $\vec{b}$ and see if $AX = B$ has a solution. Thus:

**Theorem 3.4.** Let $S$ be a collection of $m$ distinct vectors in $\mathbb{R}^n$ with the $n \times m$ matrix representation $A$. The set $S$ spans $\mathbb{R}^n$ if and only if $AX = B$ always has a solution for every $n \times 1$ matrix $B$.

**Case 1:** $m = n$ (so that we have $n$ vectors in $\mathbb{R}^n$). Then $S$ spans $\mathbb{R}^n$ if and only if $\det(A) \neq 0$ iff $\text{rref}(A) = I_n$, which means that $AX = B$ always has a solution.

**Case 2:** $m < n$ (so that we have fewer than $n$ vectors in $\mathbb{R}^n$). Now it is impossible for $S$ to span $\mathbb{R}^n$. The $n \times m$ matrix $A$ defines a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ that cannot be onto because $m < n$. Thus $AX = B$ will not always have a solution and $S$ cannot span $\mathbb{R}^n$.

**Case 3:** $m > n$ (so that we have more than $n$ vectors in $\mathbb{R}^n$). Then $S$ spans $\mathbb{R}^n$ if and only if the reduced row echelon form of $A$ does not contain a row of all 0’s (meaning that there are never any inconsistencies for a system $AX = B$.)

**Note:** You need at least $n$ vectors to span $\mathbb{R}^n$. Fewer than $n$ vectors cannot “cover” all of $\mathbb{R}^n$ which requires $n$ dimensions.
**Linear Independence**

**Definition.** Let $S = \{u_1, \ldots, u_m\}$ be a collection of $m$ distinct vectors in a vector space $V$. Then $S$ is said to be **linearly independent** if the zero vector cannot be written as a non-trivial linear combination of the vectors in $S$. In other words, if $c_1 u_1 + \ldots + c_m u_m = \vec{0}$, then each scalar coefficient $c_1, \ldots, c_m$ must be 0.

If the zero vector can be written as a non-trivial linear combination of the vectors in $S$, then $S$ is called **linearly dependent**.

**Theorem 3.5.** Let $S = \{u_1, u_2, \ldots, u_m\}$ be linearly independent. Then none of the vectors in $S$ can be the zero vector.

**Proof.** Suppose $u_j = \vec{0}$ for some $j$ with $1 \leq j \leq m$. Then we could write $1 \times u_j = \vec{0}$; which expresses $\vec{0}$ as a non-trivial linear combination of the vectors in $S$. Thus, $S$ would be linearly dependent, which is a contradiction to the assumption.

**Theorem 3.6.** Let $S$ be a collection of $m$ distinct vectors in $\mathbb{R}^n$ with the $n \times m$ matrix representation $A$. The set $S$ is linearly independent if and only if the homogeneous system $AX = 0$ has only the trivial solution of $X = 0$.

**Case 1:** $m = n$ (so that we have $n$ vectors in $\mathbb{R}^n$). Then $S$ is linearly independent if and only if $\det(A) \neq 0$ iff $\text{ref}(A) = I_n$, which means that $AX = 0$ has only the trivial solution of $X = 0$.

**Case 2:** $m > n$ (so that we have more than $n$ vectors in $\mathbb{R}^n$). Now it is impossible for $S$ to be linearly independent. The $n \times m$ matrix $A$ defines a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ that cannot be one-to-one because $m > n$. Thus $AX = 0$ will have infinite solutions and $S$ will be linearly dependent.

**Case 3:** $m < n$ (so that we have fewer than $n$ vectors in $\mathbb{R}^n$). Then $S$ is linearly independent if and only if the reduced row echelon form of $A$ reduces to the $m \times m$ identity with $n - m$ rows of 0’s (which means that $AX = 0$ has only the trivial solution of $X = 0$).

**Note:** Given $n$ vectors in $\mathbb{R}^n$, either they are linearly independent and they span (when $\det(A) \neq 0$), or they have neither property (when $\det(A) = 0$). One property holds if and only if the other property holds.

**Note further:** A set of more than $n$ vectors cannot be linearly independent in $\mathbb{R}^n$. The “extra” vectors must be “dependent” on the others. In fact, they actually will be linear combinations of other vectors in the set which makes them redundant. The following theorem formalizes the result.
**Theorem 3.7.** A set of non-zero vectors \( S = \{u_1, \ldots, u_m\} \) is linearly dependent if and only if some vector in \( S \) is a linear combination of the preceding vectors in the list.

**Proof.** Suppose \( u_{k+1} = c_1 u_1 + \ldots + c_k u_k \) (i.e., suppose that a vector in \( S \) is a linear combination of the preceding vectors in the list). Then

\[
\vec{0} = c_1 u_1 + \ldots + c_k u_k + (-1)u_{k+1};
\]

hence, \( \vec{0} \) is a non-trivial linear combination of the vectors in \( S \). Therefore, \( S \) is linearly dependent.

On the other hand, suppose that \( S \) is linearly dependent. Then there is a non-trivial linear combination such that \( \vec{0} = c_1 u_1 + \ldots + c_m u_m \). Choose the last such non-zero coefficient \( c_{k+1} \); then \( \vec{0} = c_1 u_1 + \ldots + c_{k+1} u_{k+1} \). Hence, \( -c_{k+1} u_{k+1} = c_1 u_1 + \ldots + c_k u_k \) and \( u_{k+1} = -\frac{c_1}{-c_{k+1}} u_1 + \ldots + \frac{c_k}{-c_{k+1}} u_k \). Therefore, a vector in \( S \) is a linear combination of the preceding vectors in the list. (Note: If \( c_1 \) were the last non-zero coefficient, then we would have \( \vec{0} = c_1 u_1 \), which would \( u_1 = \vec{0} \), which is not allowed by assumption.)

**Basis**

**Definition.** Let \( S = \{u_1, \ldots, u_m\} \) be a collection of \( m \) distinct vectors in a vector space \( V \). If \( S \) is linearly independent and \( S \) spans \( V \), then \( S \) is called a basis for \( V \) and \( V \) is then an \( m \)-dimensional vector space.

**Note:** Because fewer than \( n \) vectors cannot span \( R^n \) and more than \( n \) vectors cannot be linearly independent in \( R^n \), it requires exactly \( n \) vectors to form a basis for \( R^n \).

It is easy to check whether or not \( n \) vectors actually form a basis for \( R^n \):

**Theorem 3.8.** Let \( S \) be a collection of \( n \) distinct vectors in \( R^n \) with the \( n \times n \) matrix representation \( A \). Then \( S \) is a basis if and only if \( \det(A) \neq 0 \) iff \( \text{rref}(A) = I_n \).

**Dimension**

**Definition.** In general, a vector space \( V \) is called \( n \)-dimensional if it has a basis with \( n \) elements.

**Example 2.** Let \( P_n \) be the set of real polynomials having degree \( \leq n \). Then \( P_n \) is \((n+1)\)-dimensional. Indeed, consider the set of \( n+1 \) vectors \( S = \{1, x, x^2, \ldots, x^n\} \). Every polynomial in \( P_n \) is a linear combination of those in \( S \); thus, \( S \) spans \( P_n \).

Also if \( c_n x^n + \ldots + c_1 x + c_0 = 0 \), then each \( c_i \) must be 0. So the zero polynomial cannot be written as a non-trivial linear combination of the vectors in \( S \). Thus, \( S \) is linearly independent. Therefore, \( S \) is a basis for \( P_n \) having \( n+1 \) elements.
**Standard Basis**

In $\mathbb{R}^n$, let $\vec{e}_1 = (1, 0, \ldots, 0)$, $\vec{e}_2 = (0, 1, \ldots, 0)$, \ldots, $\vec{e}_n = (0, 0, \ldots, 1)$. The matrix representation $A$ is the $n \times n$ identity $I_n$; hence, $\det(A) = 1 \neq 0$. Thus, $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is a basis for $\mathbb{R}^n$ called the *standard basis*.

Every vector in $\mathbb{R}^n$ is clearly a linear combination of $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$. For instance in $\mathbb{R}^4$, $(8, -10, 12, 16) = 8(1, 0, 0, 0) + (-10)(0, 1, 0, 0) + 12(0, 0, 1, 0) + 16(0, 0, 0, 1)$.

The standard basis for $P_n$ is $S = \{1, x, x^2, \ldots, x^n\}$ as explained in Example 2.

There can be other bases as well. For example in $\mathbb{R}^3$, let $S = \{(1, 0, 0), (1, 2, 0), (1, 2, 3)\}$.

Then $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$ and $\det(A) = 6 \neq 0$; thus, $S$ is a basis for $\mathbb{R}^3$. We can write any vector in $\mathbb{R}^3$ as a linear combination of the vectors in $S$. How about $(8, 10, -12)$?

\[
\begin{pmatrix} 1 & 1 & 1 & 8 \\ 0 & 2 & 2 & 10 \\ 0 & 0 & 3 & -12 \end{pmatrix} \rightarrow \text{rref} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & -4 \end{pmatrix} \rightarrow (8, 10, -12) = 3(1, 0, 0) + 9(1, 2, 0) -4(1, 2, 3).
\]

**Note:** The set of $m \times n$ matrices $M_{m,n}$ is an $(mn)$-dimensional vector space. What is the standard basis for $M_{2,3}$?

**Theorem 3.9.** Given a basis of vectors, $S = \{u_1, u_2, \ldots, u_n\}$ for an $n$-dimensional vector space $V$, every other vector in $V$ can be written *uniquely* as a linear combination of the vectors in $S$.

**Proof.** Because $S$ is a basis for $V$, we know that $S$ spans $V$; therefore, every other vector can be written at least one way as a linear combination of the vectors in $S$.

But suppose a vector $v$ can be written two ways:

$$v = c_1u_1 + \ldots + c_nu_n = d_1u_1 + \ldots + d_nu_n$$

Then $0 = v + (-v) = v + (-1)v = (c_1 - d_1)u_1 + \ldots + (c_n - d_n)u_n$. Because $S$ is linearly independent, we must have $c_i - d_i = 0$, for $1 \leq i \leq n$. That is, $c_i = d_i$ for all $i$, and there cannot be two distinct ways to write the linear combination.
**Theorem 3.10.** Let $S = \{ \tilde{u}_1, \ldots, \tilde{u}_n \}$ be a basis for $\mathbb{R}^n$. Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a bijective linear transformation. Then $T(S) = \{ T(\tilde{u}_1), \ldots, T(\tilde{u}_n) \}$ is also a basis for $\mathbb{R}^n$.

**Proof.** Let $\tilde{v}$ be a vector in $\mathbb{R}^n$. We will show that $\tilde{v}$ is a linear combination of the vectors in $T(S)$.

Because $T$ is bijective, it has an inverse $T^{-1}$. Because $S$ is a basis, we can write $T^{-1}(\tilde{v})$ as a linear combination of the vectors in $S$: $T^{-1}(\tilde{v}) = c_1 \tilde{u}_1 + \ldots + c_n \tilde{u}_n$. Now apply $T$ and use the properties of linearity: $\tilde{v} = T(T^{-1}(\tilde{v})) = T(c_1 \tilde{u}_1 + \ldots + c_n \tilde{u}_n) = c_1 T(\tilde{u}_1) + \ldots + c_n T(\tilde{u}_n)$.

Thus, $T(S)$ spans $\mathbb{R}^n$. Because there are $n$ vectors in $T(S)$, they also must be linearly independent; hence, $T(S)$ is a basis for $\mathbb{R}^n$. 