Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a function which maps vectors from $\mathbb{R}^n$ to $\mathbb{R}^m$. Then $T$ is called a **linear transformation** if the following two properties are satisfied:

(i) $T(c \vec{u}) = cT(\vec{u})$, for all scalars $c$ and all vectors $\vec{u}$ in $\mathbb{R}^n$;

(ii) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$, for all $\vec{u}, \vec{v}$ in $\mathbb{R}^n$.

It is equivalent to say that $T(c \vec{u} + \vec{v}) = cT(\vec{u}) + T(\vec{v})$ for all scalars $c$ and all vectors $\vec{u}, \vec{v}$ in $\mathbb{R}^n$.

In order to define such a transformation $T$, we must describe its action on a point $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$. That is, we must define $T((x_1, x_2, \ldots, x_n))$ which must be a vector in $\mathbb{R}^m$ having $m$ coordinates.

In order for $T$ to be **linear**, each coordinate in $\mathbb{R}^m$ must be a linear combination of $x_1, x_2, \ldots, x_n$. That is, $T$ must have the **functional form**

$$T((x_1, x_2, \ldots, x_n)) = (a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n, a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n, \ldots, a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n)$$

**Example 1.** Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^4$ be defined by

$$T((x_1, x_2, x_3)) = (2x_1 - 2x_2 + 3x_3, x_1 + x_2 - x_3, x_1 + 4x_2, -4x_1 - 6x_2 + 3x_3).$$

Is $T$ linear? What is $T((1, 2, 3))$?

Each of the four coordinates in the range is a linear combination of the three variables $x_1, x_2, x_3$ from the domain; thus, $T$ is a linear transformation. Also, $T((1, 2, 3)) = (2 - 4 + 9, 1 + 2 - 3, 1 + 8, -4 - 12 + 9) = (7, 0, 9, -7)$.

**Example 2.** Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be defined by

$$T((x_1, x_2, x_3)) = (\sin x_1 + \cos x_2 + 4x_3, x_1 + x_2 + x_3 + 6).$$

Neither coordinate in the range is a linear combination of the domain variables $x_1, x_2, x_3$; thus, $T$ is not a linear transformation (although $T$ is still a function).

**Note:** Throughout we shall let $\vec{0}_n$ be the zero vector in $\mathbb{R}^n$ and let $\vec{0}_m$ be the zero vector in $\mathbb{R}^m$. Then $T(\vec{0}_n) = T(\vec{0}_n + \vec{0}_n) = T(\vec{0}_n) + T(\vec{0}_n)$ in $\mathbb{R}^m$. By subtracting $T(\vec{0}_n)$ from both sides, we see that $\vec{0}_m = T(\vec{0}_n)$. So it is always the case that $T(\vec{0}_n) = \vec{0}_m$ with a linear transformation.
The Matrix Representation

Every linear transformation \( T: \mathbb{R}^n \rightarrow \mathbb{R}^m \) can be represented by a unique \( m \times n \) matrix \( A \). The \( m \) rows are the coefficients from the linear combinations of the function form coordinates in \( \mathbb{R}^m \). That is, suppose \( T: \mathbb{R}^n \rightarrow \mathbb{R}^m \) is defined by

\[
T((x_1, x_2, \ldots, x_n)) =
\]

\[
(a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n, a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n, \ldots, a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n),
\]

then \( A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \). If we let \( \bar{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \), then we can write \( X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \) as an \( n \times 1 \) matrix.

The function value \( T((x_1, x_2, \ldots, x_n)) \) then can be computed by the matrix product \( AX \) which gives an \( m \times 1 \) matrix that is the function value point in \( \mathbb{R}^m \). We therefore interchangeably write \( T(\bar{x}) = AX \).

**Example 3.** Let \( T: \mathbb{R}^3 \rightarrow \mathbb{R}^4 \) be defined by

\[
T((x_1, x_2, x_3)) = (2x_1 - 2x_2 + 3x_3, x_1 + x_2 - x_3, x_1 + 4x_2, -4x_1 - 6x_2 + 3x_3).
\]

The matrix representation is the \( 4 \times 3 \) matrix \( A = \begin{pmatrix} 2 & -2 & 3 \\ 1 & 1 & -1 \\ 1 & 4 & 0 \\ -4 & -6 & 3 \end{pmatrix} \). How do we evaluate \( T((1, 2, 3)) \)? Let \( X = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \), then

\[
T((1, 2, 3)) = A \times X = \begin{pmatrix} 2 & -2 & 3 \\ 1 & 1 & -1 \\ 1 & 4 & 0 \\ -4 & -6 & 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 9 \\ -7 \end{pmatrix}
\]

\((7, 0, 9, -7)\) as coordinates in \( \mathbb{R}^4 \).

What is the action of \( T \) on the standard basis in \( \mathbb{R}^3 \)?
We see that
\[
\begin{pmatrix}
2 & -2 & 3 \\
1 & 1 & -1 \\
1 & 4 & 0 \\
-4 & -6 & 3
\end{pmatrix}
\times
\begin{pmatrix}
1 \\
0 \\
1 \\
0
\end{pmatrix} =
\begin{pmatrix}
2 \\
1 \\
1 \\
-4
\end{pmatrix},
\quad \text{and}
\begin{pmatrix}
2 & -2 & 3 \\
1 & 1 & -1 \\
1 & 4 & 0 \\
-4 & -6 & 3
\end{pmatrix}
\times
\begin{pmatrix}
0 \\
1 \\
0 \\
0
\end{pmatrix} =
\begin{pmatrix}
-2 \\
1 \\
4 \\
-6
\end{pmatrix},
\]

Thus, the columns of \( A \) are simply the action of \( T \) on the standard basis elements \((1, 0, 0), (0, 1, 0), \) and \((0, 0, 1)\). Therefore, each of the columns of \( A \) are actually elements in the range of \( T \).

**A Matrix Defines a Linear Transformation**

Reversing the process, we see that any \( m \times n \) matrix \( A \) defines a linear transformation \( T_A : \mathbb{R}^n \to \mathbb{R}^m \) by

\[
T_A(x) = AX,
\]

where \( \bar{x} = (x_1, x_2, \ldots, x_n) \) and \( X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \).

For example, given the \( 2 \times 3 \) matrix \( A = \begin{pmatrix} 3 & -2 & 8 \\ 4 & 6 & -3 \end{pmatrix} \), then \( T_A : \mathbb{R}^3 \to \mathbb{R}^2 \) is defined by

\[
T\left((x_1, x_2, x_3)\right) =
\begin{pmatrix}
3 & -2 & 8 \\
4 & 6 & -3
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
\]

\[
= \begin{pmatrix}
3x_1 - 2x_2 + 8x_3 \\
4x_1 + 6x_2 - 3x_3
\end{pmatrix}
= (3x_1 - 2x_2 + 8x_3, 4x_1 + 6x_2 - 3x_3).
\]

**Defining \( T \) by Action on the Standard Basis**

A linear transformation \( T : \mathbb{R}^n \to \mathbb{R}^m \) can also be defined by its action on the standard basis. Let \( \bar{e}_1 = (1, 0, \ldots, 0), \bar{e}_2 = (0, 1, \ldots, 0), \ldots, \bar{e}_n = (0, 0, \ldots, 1) \). If we know what \( T \) does to each basis element, then we can evaluate \( T \) at any point.

Because \( T \) is linear, we have

\[
T((x_1, x_2, \ldots, x_n)) = T(x_1 \bar{e}_1 + x_2 \bar{e}_2 + \ldots + x_n \bar{e}_n) = x_1 T(\bar{e}_1) + x_2 T(\bar{e}_2) + \ldots + x_n T(\bar{e}_n).
\]

Also, once we know the action on the standard basis, then we also know the columns of the matrix representation \( A \); thus, we can easily evaluate \( T \) at any point.
Example 4. Evaluate \( T((-2, 3, 8, -10)) \) for \( T : \mathbb{R}^4 \to \mathbb{R}^3 \) defined by

\[
T((1, 0, 0, 0)) = (-1, 2, -3) \quad T((0, 1, 0, 0)) = (2, -1, 4) \\
T((0, 0, 1, 0)) = (1, 2, 4) \quad T((0, 0, 0, 1)) = (-2, 0, 6).
\]

Solution. Because \( T \) is linear, we have

\[
T((-2, 3, 8, -10)) = -2 T((1, 0, 0, 0)) + 3 T((0, 1, 0, 0)) + 8 T((0, 0, 1, 0)) - 10 T((0, 0, 0, 1))
\]

\[
= -2 (-1, 2, -3) + 3 (2, -1, 4) + 8 (1, 2, 4) - 10 (-2, 0, 6) = (36, 9, -10).
\]

Or we can write \( A = \begin{pmatrix} 1 & 2 & 1 & -2 \\ 2 & -1 & 2 & 0 \\ -3 & 4 & 4 & 6 \end{pmatrix} \), where the columns of \( A \) are the action on the st. basis. Then

\[
T((-2, 3, 8, -10)) = \begin{pmatrix} -1 & 2 & 1 & -2 \\ 2 & -1 & 2 & 0 \\ -3 & 4 & 4 & 6 \end{pmatrix} \begin{pmatrix} -2 \\ 3 \\ 8 \\ -10 \end{pmatrix} = \begin{pmatrix} 36 \\ 9 \\ (36, 9, -10) \end{pmatrix}.
\]

One-to-One Transformations

Let \( T : \mathbb{R}^m \to \mathbb{R}^m \) be a linear transformation. Then \( T \) is one-to-one (or injective) if it is always the case that different values in the domain are assigned to different values in the range. That is, whenever \( \vec{u} \neq \vec{v} \) then \( T(\vec{u}) \neq T(\vec{v}) \). By the contrapositive, it is logically equivalent to say that whenever \( T(\vec{u}) = T(\vec{v}) \), then \( \vec{u} = \vec{v} \).

To prove that a function is one-to-one, we often assume that \( T(\vec{u}) = T(\vec{v}) \) and then argue that \( \vec{u} \) must equal \( \vec{v} \). With a linear transformation \( T \) from \( \mathbb{R}^n \) to \( \mathbb{R}^m \), we can use the following results to check if \( T \) is one-to-one:

**Theorem 2.1.** Let \( T : \mathbb{R}^n \to \mathbb{R}^m \) be a linear transformation with matrix representation \( A \). Then \( T \) is one-to-one if and only if \( T(\vec{z}) = \vec{0}_m \) has only the trivial solution of \( \vec{z} = \vec{0}_n \).

**Proof.** We know that \( T(\vec{0}_n) = \vec{0}_m \) for any linear transformation. Suppose first that \( T \) is one-to-one. If \( T(\vec{z}) = \vec{0}_m \) also, then \( \vec{z} = \vec{0}_n \) because \( T \) is one-to-one.

Now suppose that \( \vec{z} = \vec{0}_n \) is the only solution of \( T(\vec{z}) = \vec{0}_m \). To prove that \( T \) is one-to-one, assume that \( T(\vec{x}) = T(\vec{y}) \). Then \( \vec{0}_m = T(\vec{x}) - T(\vec{y}) = T(\vec{x} - \vec{y}) \) which means that \( \vec{x} - \vec{y} = \vec{0}_n \). Hence, \( \vec{x} = \vec{y} \) and \( T \) is one-to-one. QED

**Note:** Using matrix equations, Theorem 2.1 is equivalent to saying that \( T \) is one-to-one if and only if the homogeneous system \( AX = 0 \) has only the trivial solution of \( X = 0 \).
Theorem 2.2. Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with matrix representation $A$. If $m < n$, then $T$ cannot be one-to-one.

Proof. Note: Because $\mathbb{R}^n$ is a “larger” set than $\mathbb{R}^m$ when $m < n$, it should not be possible to map $\mathbb{R}^n$ to $\mathbb{R}^m$ in a one-to-one fashion.

To prove the result, observe that the matrix representation $A$ will be an $m \times n$ matrix with fewer rows than columns. Thus the last (non-zero) row of the reduced row echelon form of the homogeneous system $A^T \mathbf{x} = \mathbf{0}$ will have independent (or free) variables. Hence $A^T \mathbf{x} = \mathbf{0}$ will have infinite solutions (not just the trivial solution). By Theorem 1, $T$ cannot be one-to-one.

\[
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 1 & c_n
\end{pmatrix}
\]

Theorem 2.3. Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with matrix representation $A$. Assume $m > n$. Then $T$ is one-to-one if and only if the reduced row echelon form of $A$ reduces to the $n \times n$ identity with additional rows that are all 0.

Proof. To determine if $T$ is one-to-one, we must see if the homogeneous system $AX = \mathbf{0}$ has only the trivial solution. We consider the reduced row echelon form of the $m \times n$ matrix $A$. Because there are more rows than columns, we have two cases:

\[
\begin{pmatrix}
1 & \cdots & 0 \\
\cdot & 1 & \cdots \\
\cdot & \cdot & 1 \\
\cdot & \cdot & \cdot \\
0 & \cdots & 1 \\
0 & \cdots & 0
\end{pmatrix}
\]

the $n \times n$ identity with additional rows that are all 0

\[
\begin{pmatrix}
1 & \cdots & c_1 \\
\cdot & 1 & \cdots \\
\cdot & \cdot & c_n \\
\cdot & \cdot & \cdot \\
0 & \cdots & 0 \\
0 & \cdots & 0
\end{pmatrix}
\]

free variables remain in the last non-zero row

In the first case, $AX = \mathbf{0}$ has only the trivial solution; thus, $T$ is one-to-one. In the second case, $AX = \mathbf{0}$ has infinite solutions; thus, $T$ is not one-to-one.

Theorem 2.4. Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with matrix representation $A$. Then $T$ is one-to-one if and only if $\det (A) \neq 0$.

Proof. Because $A$ is now $n \times n$, we can say $\det (A) \neq 0$ if and only if the homogeneous system $AX = \mathbf{0}$ has only the trivial solution if and only if $T$ is one-to-one.
Example 5. Let $T: \mathbb{R}^3 \to \mathbb{R}^4$ be defined by

$$T((x_1, x_2, x_3)) = (2x_1 - 2x_2 + 3x_3, x_1 + x_2 - x_3, x_1 + 4x_2, -4x_1 - 6x_2 + 3x_3).$$

Is $T$ one-to-one?

Solution. We let $A$ be the matrix representation and look at the reduced row echelon form of homogeneous system $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 2 & -2 & 3 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 4 & 0 & 0 \\ -4 & -6 & 3 & 0 \end{bmatrix} \rightarrow \text{rref} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The only solution to $AX = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$; thus, $T$ is one-to-one. ($A$ reduces to the identity with an additional row of 0's.)

Example 6. Let $T: \mathbb{R}^4 \to \mathbb{R}^5$ be defined by

$$T((1, 0, 0, 0)) = (-1, 2, -3, 0, 1) \quad T((0, 1, 0, 0)) = (2, -1, 4, 1, 0) \quad T((0, 0, 1, 0)) = (1, 2, 4, 0, 2) \quad T((0, 0, 0, 1)) = (2, 6, 3, 2, 5).$$

Is $T$ one-to-one?

Solution. We look at the reduced row echelon form of homogeneous system $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} -1 & 2 & 1 & 2 & 0 \\ 2 & -1 & 2 & 6 & 0 \\ -3 & 4 & 4 & 3 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 1 & 0 & 2 & 5 & 0 \end{bmatrix} \rightarrow \text{rref} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that $AX = \mathbf{0}$ has infinitely many solutions; thus, $T$ is not one-to-one. (Matrix $A$ does not reduce to the identity with additional rows of 0.)

Example 7. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection about the $x$-axis. Is $T$ one-to-one?

Solution. We define $T$ by $T((x, y)) = (x, -y)$. The matrix representation is $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (which are the coefficients from the linear combinations of the function form coordinates). Because $\det(A) = -1 \neq 0$, we see that $T$ is one-to-one.
Onto Transformations

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then $T$ is onto (or surjective) if every vector $\tilde{b} \in \mathbb{R}^m$ has a pre-image in $\mathbb{R}^n$. That is, for every $\tilde{b} \in \mathbb{R}^m$ there is a vector $\tilde{v} \in \mathbb{R}^n$ such that $T(\tilde{v}) = \tilde{b}$. (The vector $\tilde{v}$ is a pre-image of $\tilde{b}$.) That is, the equation $T(\tilde{x}) = \tilde{b}$ always has a solution for every $\tilde{b} \in \mathbb{R}^m$.

We can use the following results to determine if a linear transformation is onto:

**Theorem 2.5.** Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with matrix representation $A$. Then $T$ is onto if and only if the matrix equation $AX = B$ always has a solution for every $m \times 1$ matrix $B$.

**Proof.** Because the equation $T(\tilde{x}) = \tilde{b}$ is equivalent to the matrix equation $AX = B$, the theorem is simply a matrix form re-statement of the definition of “onto.”

**Theorem 2.6.** Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with matrix representation $A$. If $m > n$, then $T$ cannot be onto.

**Proof.** Note: Because $\mathbb{R}^m$ is a “larger” set than $\mathbb{R}^n$ when $m > n$, it should not be possible for every element in $\mathbb{R}^m$ to have a pre-image by means of the function $T$.

To prove the result, observe that the matrix representation $A$ will be an $m \times n$ matrix with more rows than columns. Thus the reduced row echelon form of any system $AX = B$ will have the additional rows in $A$ become all 0. These rows will lead to inconsistencies for certain $m \times 1$ matrices $B$. Thus $AX = B$ will not always have a solution and $T$ cannot be onto.

\[
\begin{pmatrix}
1 & \ldots & c_1 \\
0 & 1 & \ldots & c_2 \\
\ldots & \ldots & 1 & \ldots \\
0 & 0 & 0 & 0 & c_m
\end{pmatrix}
\]

**Theorem 2.7.** Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with matrix representation $A$. Assume $m < n$. Then $T$ is onto if and only if the reduced row echelon form of $A$ does not contain a row of all 0’s.

**Proof.** To determine if $T$ is onto, we must see if every possible system $AX = B$ has a solution. We consider the reduced row echelon form of the $m \times n$ matrix $A$. Because there are fewer rows than columns, we have two cases:
If no rows become all 0, then there will never be an inconsistency. So $AX = B$ will always have a solution and $T$ is onto.

If a row becomes all 0, then there will be inconsistencies for some $B$ when trying to solve $AX = B$. Thus at times $AX = B$ will have no solution and $T$ will not be onto.

**Theorem 2.8.** Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transformation with matrix representation $A$. Then $T$ is onto if and only if $\det(A) \neq 0$.

**Proof.** Because $A$ is now $n \times n$, we can say $\det(A) \neq 0$ if and only if any system $AX = B$ always has a solution if and only if $T$ is onto.

**Theorem 2.9.** Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transformation. Then $T$ is either both one-to-one and onto (i.e., a bijection), or $T$ is neither one-to-one nor onto.

**Proof.** Let $A$ be the matrix representation of $T$. By Theorems 4 and 8, $T$ is one-to-one if and only if $\det(A) \neq 0$ if and only if $T$ is onto. Thus $T$ is one-to-one if and only if $T$ is onto. That is, $T$ is either both one-to-one and onto, or $T$ is neither one-to-one nor onto.

**Example 8.** Let $T_A : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be defined by the matrix representation

$$A = \begin{pmatrix} 1 & 0 & -2 & 4 \\ 1 & -2 & 1 & -3 \\ -2 & 1 & 0 & 2 \end{pmatrix}$$

where $T_A(x) = AX$. Is $T_A$ one-to-one and/or onto? Is the point $(4, -2, 6)$ in the range of $T_A$? If so, find a pre-image.

**Solution.** Because $\mathbb{R}^3$ is a “smaller” set than $\mathbb{R}^4$, $T_A$ cannot be one-to-one (Theorem 2). To determine if $T_A$ is onto, we consider the reduced row echelon form of $A$:

$$\begin{pmatrix} 1 & 0 & 0 & -1.2 \\ 0 & 1 & 0 & -0.4 \\ 0 & 0 & 1 & -2.6 \end{pmatrix}$$
Because no rows became all 0, there will never be an inconsistency. So every system $AX = B$ will have a solution; therefore, $T_A$ is onto.

Now we can say that for any vector $\tilde{b} \in \mathbb{R}^2$, there is a pre-image $\tilde{v} \in \mathbb{R}^4$ such that $T(\tilde{v}) = \tilde{b}$. (In fact we can see from $\text{rref}(A)$ that there will be infinite pre-images, which also verifies that $T_A$ is not one-to-one.)

In particular, we can solve the equation $T_A(\tilde{x}) = (4, -2, 6)$:

$$
\begin{pmatrix}
1 & 0 & -2 & 4 \\
1 & -2 & 1 & -3 \\
-2 & 1 & 0 & 2 \\
\end{pmatrix}
\rightarrow
\text{rref} 
\begin{pmatrix}
1 & 0 & 0 & -1.2 \\
0 & 1 & 0 & -0.4 \\
0 & 0 & 1 & -2.6 \\
\end{pmatrix}
= 
\begin{pmatrix}
x_1 = -4.8 + 1.2 t \\
x_2 = -3.6 + 0.4 t \\
x_3 = -4.4 + 2.6 t \\
x_4 = t \\
\end{pmatrix}
$$

If $t = 0$, then $(-4.8, -3.6, -4.4, 0)$ is one particular pre-image of $(4, -2, 6)$. That is, $T((-4.8, -3.6, -4.4, 0)) = (4, -2, 6)$.

**Example 9.** Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^4$ be the “embedding” map which places a point from $\mathbb{R}^3$ into $\mathbb{R}^4$ by adding a 0 in the 4th coordinate. Is $T$ one-to-one and/or onto?

**Solution.** Here we can define $T$ by $T((x, y, z)) = (x, y, z, 0)$. The matrix representation is the $4 \times 3$ matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. We see that $A$ is already in rref form and has a row of all 0’s. By Theorem 7, $T$ is not onto. (In particular, every element in $\mathbb{R}^4$ with a non-zero 4th coordinate will cause an inconsistency; thus, these elements will not be in the range.) However, the homogeneous system $AX = 0$ has only the trivial solution of $X = 0$; thus, $T$ is one-to-one.

**Inverses of Transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$**

When a linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is both one-to-one and onto, then we can find the inverse transformation $T^{-1}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that if $T((x_1, x_2, \ldots, x_n)) = (y_1, y_2, \ldots, y_n)$, then $T^{-1}((y_1, y_2, \ldots, y_n)) = (x_1, x_2, \ldots, x_n)$.

If $A$ is the matrix representation of the one-to-one and onto $T$, then we know that $\det(A) \neq 0$. Therefore $A^{-1}$ exists. Because $T((x_1, x_2, \ldots, x_n)) = (y_1, y_2, \ldots, y_n)$ is equivalent to $AX = Y$, we can apply the matrix inverse and say that $X = A^{-1}Y$. Thus, $A^{-1}$ must be the matrix representation of $T^{-1}$ because $(x_1, x_2, \ldots, x_n) = T^{-1}((y_1, y_2, \ldots, y_n))$. 
Example 10. Let \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) be defined by
\[
T((1, 0, 0)) = (1, 0, -1) \quad T((0, 1, 0)) = (2, 1, 2) \quad T((0, 0, 1)) = (1, 1, 4)
\]
Verify that \( T \) is a bijection. Find the function form of \( T^{-1} \).

Solution. The matrix representation of \( T \) is
\[
A = \begin{pmatrix}
1 & 2 & 1 \\
0 & 1 & 1 \\
-1 & 2 & 4
\end{pmatrix}
\]
and \( \det(A) = 1 \neq 0; \) thus \( T \) is both one-to-one and onto (i.e., a bijection). The matrix representation of \( T^{-1} \) is
\[
A^{-1} = \begin{pmatrix}
2 & -6 & 1 \\
-1 & 5 & -1 \\
1 & -4 & 1
\end{pmatrix}
\]
Thus,
\[
T^{-1}((x, y, z)) = (2x - 6y + z, -x + 5y - z, x - 4y + z).
\]
Note: \( T^{-1}((1, 0, -1)) = (1, 0, 0), \ T^{-1}((2, 1, 2)) = (0, 1, 0), \) and \( T^{-1}((1, 1, 4)) = (0, 0, 1). \)

Example 11. Let \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) be the reflection about the \( x \)-axis. Find \( T^{-1} \).

Solution. Here \( T((x, y)) = (x, -y) \) and \( A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \). Then \( A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) also. So \( T^{-1}((x, y)) = (x, -y) = T((x, y)). \) In other words, \( T \) is its own inverse. If you reflect about the \( x \)-axis, then you “undo” the process by reflecting back about the \( x \)-axis.

Null Space and Nullity

Let \( T : \mathbb{R}^n \to \mathbb{R}^m \) be a linear transformation, with \( m \times n \) matrix representation \( A \). The null space (or kernel of \( T \)) is the set of all vectors \( \bar{x} \in \mathbb{R}^n \) such that \( T(\bar{x}) = \bar{0}_m \). We know that \( T(\bar{0}_n) = \bar{0}_m \). If \( T \) is one-to-one, then no other vector gets mapped to the origin; so the null space is simply \( \{ \bar{0}_n \} \).

But if \( T \) is not one-to-one, then \( AX = 0 \) will have infinitely many solutions as will \( T(\bar{x}) = \bar{0}_m \). The set of all solutions to \( AX = 0 \) is called the null space. The nullity is the dimension of this null space and is given by the number of independent parameters \( (r, s, t, \text{ etc.}) \) used in writing the form of all solutions to \( AX = 0 \). And if \( T \) is one-to-one, then the nullity is simply 0.
Range and Rank

The range of $T$ is the set of all vectors $\tilde{y} \in \mathbb{R}^m$ for which $T(\tilde{x}) = \tilde{y}$ for some $\tilde{x} \in \mathbb{R}^n$. All of the columns in the matrix representation $A$ are actually elements in the range of $T$. The first column in $A$ is the value $T((1,0,0,\ldots,0))$, the second column in $A$ is the value $T((0,1,0,\ldots,0))$, etc. But some of these columns may be linear combinations of the previous columns. The columns in $A$ that are independent of each other create the dimensions in the range of $T$. The actual range of $T$ will be the collection of all linear combinations of these independent vectors, and the rank of $T$ is the dimension of this range.

**Example 12.** Let $T: \mathbb{R}^5 \to \mathbb{R}^5$ be defined by its action on the standard basis as

$T(\tilde{e}_1) = (1, 1, 2, 0, 1)$ \hspace{1cm} $T(\tilde{e}_2) = (0, -1, -1, -1, 1)$ \hspace{1cm} $T(\tilde{e}_3) = (0, 1, 1, 1, -1)$

$T(\tilde{e}_4) = (1, 1, 2, 0, 0)$ \hspace{1cm} $T(\tilde{e}_5) = (1, 0, 1, -1, 1)$

(a) Give the matrix representation and functional form of $T$.
(b) Is $T$ one-to-one and/or onto?
(c) Are $\tilde{b} = (1, 2, 3, 4, 5)$ and $\tilde{c} = (4, 2, 6, -2, 5)$ in range of $T$? If so, find pre-images.
(d) Find the null space and nullity. Give a basis for the null space.
(e) Find the range and rank.

**Solution.** (a) The columns of the matrix representation are the action on the standard basis elements. Thus,

$$A = \begin{pmatrix}
1 & 0 & 0 & 1 & 1 \\
1 & -1 & 1 & 1 & 0 \\
2 & -1 & 1 & 2 & 1 \\
0 & -1 & 1 & 0 & -1 \\
1 & 1 & -1 & 0 & 1
\end{pmatrix}$$

$$T((x_1, x_2, x_3, x_4, x_5)) = (x_1 + x_4 + x_5, x_1 - x_2 + x_3 + x_4, 2x_1 - x_2 + x_3 + 2x_4 + x_5, -x_2 + x_3 - x_5, x_1 + x_2 - x_3 + x_5)$$

(b) $\det(A) = 0$; thus, $A$ is neither one-to-one nor onto.

(c) $A|B = \begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 1 & 4 \\
1 & -1 & 1 & 1 & 0 & 2 & 2 \\
2 & -1 & 1 & 2 & 1 & 3 & 6 \\
0 & -1 & 1 & 0 & -1 & 4 & -2 \\
1 & 1 & -1 & 0 & 1 & 5 & 5
\end{pmatrix}$ \hspace{1cm} $\text{rref} \to \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\
0 & 1 & -1 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$

For $\tilde{b} = (1, 2, 3, 4, 5)$, we see that $AX = B$ has no solution, so $(1, 2, 3, 4, 5)$ is not in the range of $T$. However, there are infinite solutions for $\tilde{c} = (4, 2, 6, -2, 5)$. One such solution is $(3, 2, 0, 1, 0)$. That is, $\tilde{c} = (4, 2, 6, -2, 5)$ is in range of $T$ and $T((3, 2, 0, 1, 0)) = (4, 2, 6, -2, 5)$. 


span simply call it the "span" of these vectors.  In the preceding example, the null space is

Note have of just these vectors in

are the first, second, and fourth columns in the original matrix.

Thus, rank \( T \) is 3 and the range of \( T \) is the collections of all linear combinations of the three vectors \( T(\tilde{e}_1) = (1, 1, 2, 0, 1), T(\tilde{e}_2) = (0, -1, -1, -1, 1), \) and \( T(\tilde{e}_4) = (1, 1, 2, 0, 0) \), which are the first, second, and fourth columns in the original matrix.

In other words, if we look at just the original first, second, and fourth columns as vectors in \( \mathbb{R}^5 \), then any vector in the range of \( T \) can be written as a linear combination of just these three vectors after rref is performed. For \( \tilde{c} = (4, 2, 6, -2, 5) \) as in Part (c), we have \( \tilde{c} = 3T(\tilde{e}_1) + 2T(\tilde{e}_2) + 1T(\tilde{e}_4) \).

Note: Another way to label "all linear combinations of a collection of vectors" is to simply call it the "span" of these vectors. In the preceding example, the null space is \( \text{span}\{\tilde{v}_1, \tilde{v}_2\} \) and the range is \( \text{span}\{T(\tilde{e}_1), T(\tilde{e}_2), T(\tilde{e}_4)\} \).

Here is a description of the preceding transformation:

\[
T: \mathbb{R}^5 \rightarrow \mathbb{R}^5
\]

Let \( \tilde{e}_1 = (1, 0, 0, 0, 0), \tilde{e}_2 = (0, 1, 0, 0, 0), \tilde{e}_4 = (0, 0, 0, 1, 0) \).

\[
\text{Domain: } \mathbb{R}^5 \quad \text{Range: A Three Dimensional Subspace of } \mathbb{R}^5
\]

\[
\text{span}\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_4\} \quad \rightarrow \quad \text{span}\{T(\tilde{e}_1), T(\tilde{e}_2), T(\tilde{e}_4)\}
\]

\[
\oplus
\]

Two-dimensional Null Space

\[
\text{span}\{\tilde{v}_1, \tilde{v}_2\}
\]

\[
\rightarrow
\]

\[
\{0, 0, 0, 0, 0\}
\]

\( T \) is not one-to-one: Two dimensions of \( \mathbb{R}^5 \) all get mapped to \( \{0, 0, 0, 0, 0\} \).

\( T \) is not onto: The range is 3-d; only three dimensions of \( \mathbb{R}^5 \) have pre-images.
1. Let $T: R^3 \rightarrow R^3$ be defined in functional form by

$$T((x_1, x_2, x_3)) = (x_1 - 2x_2 + 2x_3, 2x_1 + x_2 + x_3, x_1 + x_2).$$

(a) Write the matrix representation $A$.
(b) Prove that $T$ is both one-to-one and onto.
(c) Evaluate $T((-4, 6, 1))$.
(d) Find a pre-image of the point $(-5, 3, 2)$;
   i.e., solve the equation $T((x_1, x_2, x_3)) = (-5, 3, 2)$.
(e) Give both the matrix representation and the functional form of $T^{-1}$.

2. Let $T: R^4 \rightarrow R^4$ be defined by

$$T((1, 0, 0, 0)) = (1, 2, 3, 1) \quad T((0, 1, 0, 0)) = (3, 6, 9, 3)$$
$$T((0, 0, 1, 0)) = (-2, -4, 1, 4) \quad T((0, 0, 0, 1)) = (4, 8, 5, 8).$$

(a) Write the matrix representation $A$.
(b) Prove that $T$ is neither one-to-one nor onto.
(c) Evaluate $T((-4, 6, 1, 2))$.
(d) Determine if the points $(5, 10, 8, 49)$ and $(6, -4, 12, 20)$ are in the range of $T$, and if so find pre-images of these points.
(e) Give the form of the null space and give the nullity of $T$. Also give a basis for the null space.
(f) Describe the range of $T$ and give the rank of $T$.
(g) Explain how the domain is divided into subspaces and how these subspaces are mapped into $R^4$ via $T$. 

1. Let \( T: R^6 \to R^4 \) be defined by
\[
T((x_1, x_2, x_3, x_4, x_5, x_6)) = (-x_1 + 2x_2 + 4x_4 + 5x_5 - 3x_6, 3x_1 - 7x_2 + 2x_3 + x_5 + 4x_6, 2x_1 - 5x_2 + 2x_3 + 4x_4 + 6x_5 + x_6, 4x_1 - 9x_2 + 2x_3 - 4x_4 - 4x_5 + 7x_6)
\]
(a) Give the matrix representation of \( T \).
(b) Determine and explain whether or not \( T \) is onto.
(c) Describe the range of \( T \) and give the rank of \( T \).
(d) Determine and explain whether or not \( T \) is one-to-one.
(e) Give the form of the null space and give the nullity of \( T \). Also give a basis for the null space.
(f) Explain how the domain is divided into two subspaces and how these subspaces are mapped into \( R^4 \) via \( T \).

2. Let \( T: R^5 \to R^3 \) defined by matrix representation
\[
A = \begin{pmatrix}
1 & 4 & 5 & 0 & 9 \\
3 & -2 & 1 & 0 & -1 \\
2 & 3 & 5 & 1 & 8
\end{pmatrix}
\]
(a) Give the function form of \( T \).
(b) Determine and explain whether or not \( T \) is onto.
(c) Describe the range of \( T \) and give the rank of \( T \).
(d) Determine if \((3, 2, 1)\) is in the range of \( T \). If so, give a pre-image.
(e) Determine and explain whether or not \( T \) is one-to-one.
(f) Give the form of the null space and give the nullity of \( T \). Also give a basis for the null space.

3. Let \( T: R^3 \to R^4 \) be defined by
\[
T((1, 0, 0)) = (3, 1, 0, 1) \quad T((0, 1, 0)) = (1, 2, 1, 1) \quad T((0, 0, 1)) = (-1, 0, 2, -1)
\]
(a) Give the matrix representation of \( T \).
(b) Evaluate \( T((1, 2, 3)) \).
(c) Determine and explain whether or not \( T \) is onto.
(d) Describe the range of \( T \) and give the rank of \( T \).
(e) Determine if \((1, 0, 0, 0)\) is in the range of \( T \). If so, give a pre-image.
(f) Determine and explain whether or not \( T \) is one-to-one.
(g) Give the form of the null space and give the nullity of \( T \). Also give a basis for the null space.