Adjustments to the Test Statistic

I. Arbitrary Measurements and \( \sigma \) is Unknown: We can replace \( \sigma \) with the sample deviation \( S \) provided we have a large enough sample so that \( S \) is a decent approximation of \( \sigma \). So now the test statistic is \( z = \frac{\bar{x} - M}{S / \sqrt{n}} \), and we still reject \( H_0 \) if \( z \) is too large or too small in comparison with the \( z \)-scores. The associated \( P \)-value probabilities are still computed with the standard normal distribution curve. (Z-Test)

II. Normally Distributed Measurements: There is no change if \( \sigma \) is known; but if \( \sigma \) is unknown, then we again replace \( \sigma \) with \( S \). However now the possible values of \( \frac{\bar{x} - M}{S / \sqrt{n}} \) follow a \( t(n-1) \) distribution. This value \( t = \frac{\bar{x} - M}{S / \sqrt{n}} \) is now called the \( t \)-statistic and the associated \( P \)-value probabilities are computed with the \( t(n-1) \) distribution curve rather than the standard normal curve. These calculations can be made with the T-Test screen.

Example 1. Here are the monthly fees (to the nearest in dollar) paid by a random sample of users of commercial ATM services. The fees of all users are found to be normally distributed.

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Do the data give good reason to think that the mean cost for all ATM users (i) is more than $16? (ii) differs from $20 per month?

Solution. (i) We test \( H_0: \mu = 16 \) versus \( H_a: \mu > 16 \). First, enter the data into a list, say list L1. Then bring up the T-Test screen from the STAT TESTS menu. Set the Inpt to Data, adjust the List and alternative, and calculate.

We obtain a \( t \)-statistic of 2.532 and a \( P \)-value of 0.01. If the true mean cost were equal to $16, then there would be only a 1% chance of obtaining an \( \bar{x} \) of $21.25 or higher with a random sample of size 20. This low \( P \)-value give us evidence to reject \( H_0 \) in favor of the alternative that \( \mu > 16 \).

(ii) We now test \( H_0: \mu = 20 \) vs. \( H_a: \mu \neq 20 \). If \( \mu = 20 \) were true, then there would be a 55.37% chance of obtaining an \( \bar{x} \) as far away (in either direction) as 21.25 with \( n = 20 \). No evidence to reject \( H_0 \).
III. Population Proportion: We now are analyzing a proportion $p$ in the following manner:

A null hypothesis of $H_0: p = P$ versus one of three alternative hypotheses

- $H_a: p < P$ (left-sided)
- $H_a: p > P$ (right-sided)
- $H_a: p \neq P$ (two-sided)

Assuming $H_0: p = P$ is true, then the true population standard deviation is $\sigma = \sqrt{P(1 - P)}$; thus, the $z$-statistic is $z = \frac{p - P}{\sqrt{P(1 - P)} \sqrt{n}}$. We again reject $H_0$ if $z$ is too large or too small in comparison with the $z$-scores, and the associated $P$-value probabilities are computed with the standard normal distribution curve. These calculations can be made with the 1–PropZTest screen.

Example 2. In a recent survey, 249 out of 930 new motorcycle purchases were Harley Davidson. Let $p$ be the true proportion of all purchased motorcycles that are Harleys. (i) Test whether $p$ equals 0.30 with a one-sided alternative. (ii) Test whether $p$ equals 0.25 with a two-sided alternative.

Solution. We first note that $\bar{p} = 249/930 \approx 0.2677$.

(i) $H_0: p = 0.30$ versus $H_a: p < 0.30$

If $p = 0.30$ were true, then we have only a 1.59% chance of obtaining a $\bar{p}$ of 0.2677 or lower with a sample of size 930. Reject $H_0$.

(ii) $H_0: p = 0.25$ versus $H_a: p \neq 0.25$

If $p = 0.25$ were true, then we have a 21.15% chance of obtaining a $\bar{p}$ as far away (in either direction) as 0.2677 with $n = 930$, which is not enough evidence to reject $H_0$.

Explanations in terms of test stat with $\alpha = 0.05$ level of significance:

(i) For an alternative of $H_a: p < 0.30$ at $\alpha = 0.05$ level of significance, the $z$-score that creates 0.05 probability at the left-tail is $-1.645$. The test stat is $z = \frac{\bar{p} - 0.30}{\sqrt{0.30 \times 0.70}} = \frac{-2.1467}{\sqrt{930}}$, which beyond $-1.645$; thus, $\bar{p} = 0.2677$ is too low and gives us significant evidence to reject $p = 0.30$.

(ii) For a two-sided alternative at $\alpha = 0.05$, the $z$-scores that create 0.05 total probability at the tails are $\pm 1.96$. The test stat is now 1.2495 which is within the $z$-scores. So $\bar{p} = 0.2677$ is not too far away and we do not have evidence to reject $p = 0.25$. 
Practice Exercises

1. A Gallop Poll on energy use found that 225 of 512 randomly selected adults favored increasing the use of nuclear power. Does this poll give good evidence that less than half of all adults favor increased nuclear power? State hypotheses, give the test statistic and $P$-value, and state your conclusion.

2. A random sample of 1048 high school students found that 692 had a TV in their room at home. Let $p$ be the true proportion of high school students with a TV in their room. Use the survey data to test the following hypotheses:

   (a) $H_0: p = 0.60$ versus $H_a: p > 0.60$
   (b) $H_0: p = \frac{2}{3}$ versus $H_a: p \neq \frac{2}{3}$

3. A sample of 44 great white sharks yielded a mean length of 15.59 ft with a sample deviation of 2.55. Is there significant evidence to reject the claim “Great white sharks average 20 feet in length?”

4. Below are a sample of measurements (in millimeters) of a critical dimension on a sample of auto engine crankshafts. The measurements are assumed to be normally distributed. Do these data give evidence that the process mean differs from 224 mm?

   | 224.120 | 224.001 | 224.017 | 223.982 | 223.989 | 223.961 |
   | 223.960 | 224.089 | 223.987 | 223.976 | 223.902 | 223.980 |
   | 224.098 | 224.057 | 223.913 | 223.999 |         |         |
1. We test $H_0: p = 0.50$ versus the alternative $H_a: p < 0.50$ using the 1-PropZTest screen. We note that $\overline{p} = 225/512 = 0.4394$.

The $z$ statistic is given as $-2.74$ and the $P$-value is 0.00307. If $p = 0.50$ were true, then there would be only about 0.003 probability of obtaining a $\overline{p}$ of 0.4394 or lower with a sample of size 512. This low $P$-value gives strong evidence to reject $H_0$ in favor of the alternative that $H_a: p < 0.50$. (Also, $z = -2.74$ is in the left-side rejection region beyond $-1.645$; thus $\overline{p} = 0.4394$ is too low and we can reject $p = 0.50$.)

2. Note that $\overline{p} = 692/1048 \approx 0.66$.
   
   (a) $H_0: p = 0.60$ versus $H_a: p > 0.60$
   If $p = 0.60$ were true, then there would be only about 0.000337 probability of obtaining a $\overline{p}$ of 0.66 or higher with a sample of size 1048. We have strong evidence to reject $H_0$.

   (b) $H_0: p = 2/3$ versus $H_a: p \neq 2/3$
   If $p = 2/3$ were true, then there would be a 66.22% chance of obtaining a $\overline{p}$ as far away (in either direction) as 0.66 with a sample of size 1048. There is no evidence to reject $H_0$.

3. T-Test $H_0: \mu = 20$ versus $H_a: \mu < 20$.
   We obtain a $P$-value of 0. If $\mu = 20$ were true, then there would be no chance of obtaining an $\overline{x}$ of 15.59 or lower with a sample of size 44; we have significant evidence to reject $H_0$.

4. We test $H_0: \mu = 224$ versus $H_a: \mu \neq 224$. First, enter the data into a list, say list L1. Then bring up the T-Test screen from the STAT TESTS menu. Set the Inpt to Data, adjust the List and alternative, and calculate or draw.

   We obtain a $t$ statistic of 0.1254 and a $P$-value of 0.9019. If $\mu = 224$ were true, then there would be over a 90% chance of obtaining a sample mean as far away as $\overline{x} = 224.0019375$ with a random sample of size 16. This very high $P$-value means that we should accept $H_0$. 

Dr. Neal, WKU