Suppose we are considering two distinct populations $\Omega_1$ and $\Omega_2$ and we wish to study the difference $p_1 - p_2$ between the proportions of those in each population who have a certain designation. We first wish to test whether the proportions are equal:

<table>
<thead>
<tr>
<th>Hypothesis</th>
<th>Description</th>
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<tbody>
<tr>
<td>$H_0$: $p_1 = p_2$</td>
<td>with one-sided alternative</td>
</tr>
<tr>
<td>$H_a$: $p_1 &lt; p_2$</td>
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<tr>
<td>$H_a$: $p_1 \neq p_2$</td>
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</table>

We base our decision for each test on the difference in sample proportions $\bar{p}_1 - \bar{p}_2$ from large independent random samples of sizes $n_1$ from $\Omega_1$ and $n_2$ from $\Omega_2$.

Here we let $\bar{p}_1 = \frac{x_1}{n_1}$ and $\bar{p}_2 = \frac{x_2}{n_2}$, where $x_1$ and $x_2$ are the number of “Yes” responses in the respective samples. Under the assumption that the true proportions are equal, we can pool the sample data to obtain an estimate for this common proportion:

$$\hat{p} = \frac{x_1 + x_2}{n_1 + n_2},$$

which is the proportion of the total number of “Yes” responses in from both samples combined. We use this pooled sample proportion $\hat{p}$ and the individual sample proportions $\bar{p}_1$ and $\bar{p}_2$ to define a test statistic $z$ by

$$z = \frac{(\bar{p}_1 - \bar{p}_2)}{\sqrt{\frac{\bar{p}(1 - \bar{p})}{n_1} + \frac{\hat{p}(1 - \hat{p})}{n_2}}},$$

For large samples, the test statistic follows an approximate standard normal distribution $Z \sim N(0,1)$. We then compute the left and right tail probability values created by the test statistic, $P(Z \leq z)$ and $P(Z \geq z)$, to give the $P$-values for the one-sided tests. The TI-83/84 command $\text{2-PropZTest}$ will perform these calculations.

**Example 1.** In a study of math students at North Carolina State University, the backgrounds of students successful in entry-level courses were checked. For students from city backgrounds, 132 out of 200 succeeded. For students from rural backgrounds, 84 out of 120 succeeded.

Is there evidence that the proportion of successful students is higher for one group of these two types of students? State hypotheses, give the $P$-value of a test, and state your conclusion.
Solution. Let $p_1$ be the true proportion of city students who succeed in the course, and let $p_2$ be the true proportion of all rural students who succeed. Then $\bar{p}_1 = \frac{132}{200} = 0.66$, $\bar{p}_2 = \frac{84}{120} = 0.70$, and $\bar{p}_1 - \bar{p}_2 = -0.04$. Also, $\hat{p} = \frac{216}{320} = 0.675$. Because $\bar{p}_1 < \bar{p}_2$, we test the claim $H_0: p_1 = p_2$ versus the alternative $H_a: p_1 < p_2$. To do so, we use the 2-PropZTest screen from the STAT TESTS menu:

\[ z = \frac{(\hat{p}_1 - \hat{p}_2)}{\sqrt{\frac{\hat{p}(1 - \hat{p})}{n_1} + \frac{\hat{p}(1 - \hat{p})}{n_2}}}. \]

\[ z = \frac{0.66 - 0.70}{\sqrt{\frac{0.675 \times 0.325}{200} + \frac{0.675 \times 0.325}{120}}} = -0.7396 \]

With a 5% level of significance for a one-sided test, the left-tail $z$-score is $-1.645$. The test statistic is not in the rejection region beyond $-1.645$; thus, $\bar{p}_1 - \bar{p}_2$ is not too small and does not give us enough evidence to reject $H_0$.

Alternate Solution. The test statistic is displayed in the output and is given by

\[ z = \frac{(\hat{p}_1 - \hat{p}_2)}{\sqrt{\frac{\hat{p}(1 - \hat{p})}{n_1} + \frac{\hat{p}(1 - \hat{p})}{n_2}}}. \]

\[ z = \frac{0.66 - 0.70}{\sqrt{\frac{0.675 \times 0.325}{200} + \frac{0.675 \times 0.325}{120}}} = -0.7396 \]

Testing for a Non-Zero Difference

We also may wish to test whether the difference in proportions $p_1 - p_2$ is equal to some fixed value other than 0:

1. $H_0: p_1 - p_2 = P$ with one-sided alternative $H_a: p_1 - p_2 > P$.
2. $H_0: p_1 - p_2 = P$ with one-sided alternative $H_a: p_1 - p_2 < P$.
3. $H_0: p_1 - p_2 = P$ with two-sided alternative $H_a: p_1 - p_2 \neq P$.

In this case, we define the test statistic by

\[ z = \frac{(\hat{p}_1 - \hat{p}_2) - P}{\sqrt{\frac{\hat{p}_1(1 - \hat{p}_1)}{n_1} + \frac{\hat{p}_2(1 - \hat{p}_2)}{n_2}}}. \]

which follows an approximate standard normal distribution $Z \sim N(0,1)$ for large sample sizes. We then can compute the $P$-value created by the test statistic, or just compare the test statistic with the rejection region.
Example 2. A poll commissioned by the Center on Addiction and Substance Abuse at Columbia University found that 304 out of 400 youths and 1340 of 2000 adults believed that popular culture encourages drug use. Let $p_1$ and $p_2$ denote the true proportions of all youths and all adults having this belief.

Is there strong evidence that the proportion of youths with this belief is more than 5 percentage points higher than the proportion of adults with the belief?

Solution. Here $\bar{p}_1 = 304/400 = 0.76$ (for youths) and $\bar{p}_2 = 1340/2000 = 0.67$ (for adults). So the difference is $\bar{p}_1 - \bar{p}_2 = 0.09$. We shall test $H_0: p_1 - p_2 = 0.05$ versus the one-sided alternative $H_a: p_1 - p_2 > 0.05$ with a 5% level of significance.

The test statistic is

$$z = \frac{(\bar{p}_1 - \bar{p}_2) - P}{\sqrt{\frac{\bar{p}_1(1-\bar{p}_1)}{n_1} + \frac{\bar{p}_2(1-\bar{p}_2)}{n_2}}} = \frac{0.09 - 0.05}{\sqrt{\frac{0.76 \times 0.24}{400} + \frac{0.67 \times 0.33}{2000}}} \approx 1.68$$

and $P(Z \geq 1.68) = 0.0465$ using normalcdf(1.68,1E99,0,1). The small $P$-value of 0.0465 gives statistical evidence to reject $H_0$. If $p_1 - p_2 = 0.05$ were true, then there would be only a 4.65% chance of obtaining $\bar{p}_1 - \bar{p}_2$ of 0.09 or higher with samples of these sizes. Thus we can assert that the difference between youths and adults with this belief is more than 5 percentage points.

Moreover, the test stat of 1.68 is beyond the right-side rejection region endpoint of 1.645. Thus, $\bar{p}_1 - \bar{p}_2 = 0.09$ is too large and gives enough evidence to reject $H_0$.

Exercises

1. A randomized experiment was conducted to test whether adding medicine to fight depression increases the effectiveness of nicotine patches in helping smokers to quit. After a year, 87 out of 245 smokers in the patch-plus-drug group had quit compared to 40 out of 244 smokers in the patch-only group.

(a) Is there significant evidence that the medicine increases the success rate?
(b) Does the data support the claim that the medicine increases effectiveness by 20 percentage points?

For each, state hypotheses, calculate a test statistic, give the $P$-value, and state your conclusion.

2. In a statewide poll, 102 out of 510 Kentuckians rated themselves as die-hard UK basketball fans. At the same time, a survey of Western’s students found that 63 out of 300 students rated themselves as die-hard UK basketball fans.

Test the hypothesis that the proportion of die-hard UK fans around the state was the same as the proportion of die-hard fans at WKU at that time. State hypotheses, calculate a test statistic, give the $P$-value, and state your conclusion.
Solutions

1. Let $p_1$ be the true proportion of smokers who would quit after using the patch-plus-drug treatment, and let $p_2$ be the true proportion of smokers who would quit using only the patch. Here $\bar{p}_1 = \frac{87}{245} \approx 0.3551$ and $\bar{p}_2 = \frac{40}{244} \approx 0.164$. Then $p_1 < p_2 \approx 0.1911$ and $\hat{p} = \frac{87 + 40}{245 + 244} = \frac{127}{489} \approx 0.26$.

(a) We will test the claim $H_0: p_1 = p_2$ with the alternative $H_a: p_1 > p_2$.

We obtain a $P$-value of about 0.000000717 from a test statistic of:

$$z = \frac{(0.3551 - 0.164) - 0.20}{\sqrt{\frac{0.26 \times 0.74}{245} + \frac{0.26 \times 0.74}{244}}} \approx 4.82.$$ 

Thus, if the true proportions of smokers who would quit under these treatments were equal, then there would be almost no chance of $\bar{p}_1 - \bar{p}_2$ being 0.1911 or higher with samples of these sizes. This very low $P$-value gives significant evidence to reject $H_0$ and conclude that the patch-plus-drug treatment is more effective.

Moreover, the test stat of 4.82 is beyond the right-side rejection region endpoint of 1.645. Thus, $\bar{p}_1 - \bar{p}_2 = 0.1911$ is too large and gives enough evidence to reject $H_0$.

(b) Note again that $\bar{p}_1 - \bar{p}_2 \approx 0.1911$. We now test the hypothesis $H_0: p_1 - p_2 = 0.20$ versus the alternative $H_a: p_1 - p_2 < 0.20$. The test statistic is

$$z = \frac{(\bar{p}_1 - \bar{p}_2) - 0.20}{\sqrt{\frac{0.3551 \times (1 - 0.3551)}{245} + \frac{0.164 \times (1 - 0.164)}{244}}} \approx -0.2283228$$

The test stat of −0.2283 is not in the rejection beyond the left-side endpoint of −1.645. Thus, $\bar{p}_1 - \bar{p}_2 = 0.1911$ is not too small and we cannot reject $H_0: p_1 - p_2 = 0.20$.

2. Let $p_1$ be the true proportion of statewide UK fans, and let $p_2$ be the true proportion UK fans at Western. Here $\bar{p}_1 = \frac{102}{510} \approx 0.2$ and $\bar{p}_2 = \frac{63}{300} = 0.21$. So $\hat{p} = \frac{102 + 63}{510 + 300} = \frac{165}{810} \approx 0.2037$. We will test the hypothesis $H_0: p_1 = p_2$ versus the alternative $H_a: p_1 \neq p_2$.

We obtain a $P$-value of about 0.733 from a test statistic of −0.341245. Thus, if $p_1 = p_2$ were true, then there would still be a 73.3% chance of $\bar{p}_1$ and $\bar{p}_2$ differing by 0.01 with samples of these sizes. This high $P$-value gives support to the null hypothesis.