Given a polynomial-type function, we will analyze the graph to find the relative extreme values and to determine where the graph changes concavity.

**Algebraic Steps for Relative Extrema, Increasing/Decreasing**

1. Graph the function globally with your calculator. If necessary, zoom in to obtain a better view of where the function changes increasing/decreasing and/or concavity.

2. Evaluate $f'(x)$. Find the critical points as follows: (i) Solve $f'(x) = 0$. (ii) Determine other points in the domain of $f$ where $f'(x)$ is undefined.

3. Determine which if any of these critical points yield a relative maximum or a relative minimum. A relative max occurs at $c$ if $f'$ changes from positive to negative at $c$. A relative min occurs at $c$ if $f'$ changes from negative to positive at $c$. Evaluate the function at these points to find the function values of the relative extrema.

   The approximate numerical values of the critical points and the relative extreme values also can be found with the built-in “max” and “min” commands. But for best precision, it is best to use the “solve” command to solve $f'(x) = 0$.

4. If there is a relative maximum at some point $c$, then the graph of $f$ must change from increasing to decreasing at this point. If there is a relative minimum at some point $c$, then the graph of $f$ must change from decreasing to increasing at this point.

   Note that when $f'(x) > 0$, then the function $f$ is increasing. When $f'(x) < 0$, then the function $f$ is decreasing.

**Concavity and Inflection Points**

When $f''(x) > 0$, then the function $f$ is concave up. When $f''(x) < 0$, the function $f$ is concave down. The points on the graph where $f$ changes concavity are called inflection points.

5. Evaluate $f''(x)$. Determine where $f''(x) = 0$ and other points in the domain of $f$ where $f''(x)$ is undefined. From the graph of $f''$, determine if $f''$ is changing from positive to negative (or vice versa) at these points. If so, then the original function $f$ is changing concavity. If $f$ changes concavity at point $d$, then we say that an inflection point occurs at $d$, and the point $(d, f(d))$ on the graph is called an inflection point.

**Example 1**. Analyze the graph of $f(x) = x^4 - 2x^2 + 6x - 6$. Find all relative extreme values and state where they occur, and state where the inflection points occur. State where the graph is increasing/decreasing and where it is concave up/concave down.
Solution. We first shall graph $f(x)$, $f'(x)$, and $f''(x)$ in $Y1$, $Y2$, and $Y3$.

\[ f(x) = x^4 - 2x^2 + 6x - 6 \quad f'(x) = 4x^3 - 4x + 6 \quad f''(x) = 12x^2 - 4 \]

We now solve where $f'(x) = 0$ to obtain the critical points. From the graph of $f'(x)$ we see that there is only one solution to $f'(x) = 0$. This point is where the original graph of $f(x)$ is obtaining its minimum.

Since there is no algebraic solution to $4x^3 - 4x + 6 = 0$, we shall use either the “minimum” command on $f(x)$ in $Y1$, or the “zero/root” command on $f'(x)$ in $Y2$. Each gives an approximate solution to the critical point. We also can use the “solve” command to find where $Y2$ equals 0.

Thus, the minimum value occurs at $x \approx -1.431127144$ and the minimum value is $y \approx -14.4882$. There are no other relative minimum or relative maximum values, and there is no absolute maximum.

The graph of $f$ is decreasing on $(-\infty, -1.431127144)$, which is also where $f'(x) < 0$. The graph of $f$ is increasing on $(-1.431127144, \infty)$, which is also where $f'(x) > 0$.

Next we find the roots of $f''(x)$. From its graph, we can see that there are two solutions to $f''(x) = 0$. Solving $12x^2 - 4 = 0$, gives $x^2 = 1/3$ and $x = \pm 1/\sqrt{3}$.

Since $f''(x) < 0$ within $(-1/\sqrt{3}, 1/\sqrt{3})$ and $f''(x) > 0$ outside of this interval, we see that the original function $f$ is concave down on $(-1/\sqrt{3}, 1/\sqrt{3})$ and $f$ is concave up on $(-\infty, -1/\sqrt{3}) \cup (1/\sqrt{3}, \infty)$. The inflection points occur where $f$ changes concavity at $x = \pm 1/\sqrt{3}$.

Example 2. Analyze the graph of $f(x) = \frac{1}{5}x^5 - \frac{1}{2}x^4 - x^3 + 12$. 

Solution. Again we first graph \( f(x) \), \( f'(x) \), and \( f''(x) \) in \( Y_1 \), \( Y_2 \), and \( Y_3 \).

We now solve where \( f'(x) = 0 \) to obtain the critical points. Solving the equation:
\[
x^4 - 2x^3 - 3x^2 = 0
\]
gives \( x^2(x^2 - 2x - 3x) = 0 \) and \( x^2(x + 1)(x - 3) = 0 \). Thus, we have three critical points: \( x = 0 \), \( x = -1 \), and \( x = 3 \).

From the graph of \( f'(x) \) we see that \( f'(x) > 0 \) if \( x < -1 \) or if \( x > 3 \), and \( f'(x) < 0 \) when \( -1 < x < 3 \). Thus, \( f \) is increasing on \(( -\infty, -1) \), then decreasing on \(( -1, 3) \), then increasing on \(( 3, \infty) \).

Because \( f \) changes from increasing to decreasing at \( x = -1 \), then \( f(-1) = 12.3 \) is a relative maximum. Because \( f \) changes from decreasing to increasing at \( x = 3 \), then \( f(3) = -6.9 \) is a relative minimum. (There is no absolute max or absolute min.)

Now we find the roots of \( f''(x) \) to determine concavity. Solving \( f''(x) = 0 \) gives
\[
4x^3 - 6x^2 - 6x = 0,
\]
then \( 2x(2x^2 - 3x - 3) = 0 \). Using the quadratic formula, the solutions are \( x = 0 \), \( x = (3 - \sqrt{33}) / 4 \approx -0.68614 \), and \( x = (3 + \sqrt{33}) / 4 \approx 2.18614 \). Since \( f''(x) \) changes positive/negative at each of these points, then the inflection points occur at these values of \( x = 0 \), \( x \approx -0.68614 \), and \( x \approx 2.18614 \).

From observing where \( f''(x) < 0 \) and where \( f''(x) > 0 \), we see that \( f \) is concave down approximately on \(( -\infty, -0.68614) \), then concave up on \(( -0.68614, 0) \), then concave down approximately on \(( 0, 2.18614) \), then concave up on \(( 2.18614, \infty) \).

Note: You can also find the approximate values of the critical points, relative extrema, and inflection points by using the built-in “min/max” commands and the built-in “zero/root” commands.

Example 3. Analyze the graph of \( f(x) = 9x^{2/3} - x^2 + 8 \).
Solution. Below are the graphs of $f(x)$, $f'(x)$, and $f''(x)$:

- $f(x) = 9x^{2/3} - x^2 + 8$
- $f'(x) = \frac{6}{x^{1/3}} - 2x$
- $f''(x) = -2\frac{2}{x^{4/3}} - 2$

Recall in general that there are two types of critical points: (i) Whenever $f'(x) = 0$ and (ii) Whenever $f'(x)$ is undefined but $f(x)$ is defined.

(i) In this case, $f'(x) = 0$ has two solutions: $\frac{6}{x^{1/3}} - 2x = 0$ gives $\frac{6}{x^{1/3}} = 2x$, then $3 = x^{4/3}$. Cubing both sides gives $x^4 = 27$. Hence, $x = \pm 27^{1/4}$. Because $f'(x)$ changes from positive to negative at both of these points, $f$ is changing from increasing to decreasing at both points giving relative maximums. In fact, an absolute maximum occurs at each point and is given by $f(\pm 27^{1/4}) \approx 18.3923$.

(ii) Another critical point occurs at $x = 0$ which is where $f'(x)$ is undefined but $f(x)$ is defined. At $x = 0$, $f'$ changes from negative to positive, which means that $f$ is changing from decreasing to increasing yielding a relative minimum value of $f(0) = 8$.

So $f$ is increasing on $(-\infty, -27^{1/4})$, then decreasing on $(-27^{1/4}, 0)$, then increasing on $(0, 27^{1/4})$, then decreasing on $(27^{1/4}, \infty)$.

For concavity, we now need to check where $f''(x) = 0$ and where $f''(x)$ is undefined but $f'(x)$ is defined. In the first case, $f''(x) = 0$ gives $-2\frac{2}{x^{4/3}} - 2 = 0$, then $-2 = \frac{2}{x^{4/3}}$ and $x^{4/3} = -1$. But this equation has no solution because $x^{4/3} \geq 0$ for all $x$.

Next, $f''(x)$ is undefined at $x = 0$; but $f''$ does not change signs at $x = 0$ because $f''(x) < 0$ for all $x \neq 0$. Thus, $f$ is always concave down and there are no inflection points.