Let $f(x)$ be a differentiable function with an observable root $c$ (i.e., $f(c) = 0$). If we cannot solve for $c$ algebraically, then we must approximate $c$ or solve for it numerically. *Newton’s Method* gives an iterative process for approximating $c$.

We first let $x_0$ be an initial estimate of $c$ that can be obtained by observation after graphing $f$. Next consider the graph of the tangent line to $f$ at point $(x_0, f(x_0))$. Provided this line does not have zero slope, then it will intersect the $x$-axis. The root $x_1$ of the tangent line becomes a better approximation of $c$.

![Graph of function and tangent line](image)

Then we repeat the process using $x_1$ as the estimate of $c$. That is, we find the tangent line to $f$ at point $(x_1, f(x_1))$, and then find the root $x_2$ of this tangent line. Then $x_2$ is an even better approximation of $c$.

The equation of the initial tangent line through point $(x_0, f(x_0))$ is given by $y = f'(x_0)(x - x_0) + f(x_0)$. Letting $y = 0$ and solving for $x$ gives us the root of this tangent line which is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$ 

Then $c \approx x_1$. Repeating the process with $x_1$, we obtain the root of the second tangent line as

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$ 

Now $c \approx x_2$. In general, if $x_n$ is an estimate of $c$, then the next better estimate is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$ 

We start with $x_0$ as the initial estimate of $c$ and repeat the procedure over and over until the process “converges.”
Example 1. Perform two iterations of Newton’s Method “by hand” to estimate the positive solution of the equation \( 9\sqrt{x} - x^2 = 0 \).

Solution. Clearly \( x=0 \) is one solution. From the graph of the function \( f(x) = 9\sqrt{x} - x^2 \), it appears that there is another solution between 4 and 5. We shall let \( x_0 = 4 \) be the initial estimate because 4 appears to be the closest integer approximation of the root. Next, \( f'(x) = \frac{9}{2\sqrt{x}} - 2x \), and the next approximation is given by

\[
x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 4 - \frac{f(4)}{f'(4)} = 4 - \frac{2}{-5.75} = \frac{100}{23} \approx 4.347826.
\]

The second iteration already becomes messy as is usually the case. This second iteration gives the next approximation as

\[
x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{100}{23} - \frac{f(\frac{100}{23})}{f'(\frac{100}{23})} \approx 4.326825.
\]

So an approximate solution of \( 9\sqrt{x} - x^2 = 0 \) is \( c \approx 4.326825 \).

Even with a calculator, it is tedious to compute \( x_2 \) without error. But there is a better way to use your calculator. To implement Newton’s Method on the TI-84, use the following steps:

**Step 1:** Store \( f(x) \) into the \( Y_1 \) screen and store the derivative \( f'(x) \) into \( Y_2 \).

Rather than evaluating the derivative as a function, you instead can use the numerical derivative command: \( Y_2 = \text{nDeriv}(Y_1, X, X) \)

**Note:** When trying to solve an equation such as \( g(x) = a \), then re-write the equation as \( g(x) - a = 0 \) and use \( g(x) - a \) as the function \( f(x) \).

**Step 2:** Graph \( f(x) \) to get an eyeball approximation of the root (or roots). (De-select \( Y_2 \) so that only \( Y_1 \) is graphed.) From the graph of \( f(x) \), pick an initial guess \( x_0 \) of the root (usually the closest integer).

**Step 3:** Store the value of \( x_0 \) to \( X \) on the calculator: \( x_0 \rightarrow X \)

**Step 4:** Perform Newton’s Method until the process converges.

Enter \( X - Y_1(X)/Y_2(X) \rightarrow X \)

This command computes the next approximation and re-stores the value to \( X \).

Press 2nd ENTER ENTER to get the next approximation. Keep pressing 2nd ENTER ENTER until the process converges.
**Example 2.** Use Newton’s Method on your calculator to estimate a solution of the equation \(x^3 - 3x^2 + 4x = 10\). Use the nearest integer as the initial approximation.

**Solution.** We let \(f(x) = x^3 - 3x^2 + 4x - 10\) and \(f'(x) = 3x^2 - 6x + 4\). We enter \(f\) and \(f'\) into \(Y_1\) and \(Y_2\), then estimate a root of \(f\). From the graph of \(f\), it appears that the root is \(c \approx 3\). So we let \(x_0 = 3\) and proceed with Newton’s Method. We find that \(x_1 = 2.846153846\) and \(x_2 = 2.83382669\). We continue three more times. We find that \(x_4 = x_5\) (up to 9 decimal places), so the process has converged and we may stop. We conclude that \(c \approx 2.833750958\) is an approximate solution to \(x^3 - 3x^2 + 4x = 10\).

**Note:** Using \(Y_2 = \text{nDeriv}(Y_1, X, X)\) often gives slightly different values for \(x_1\) and \(x_2\), but will still yield the same final value.