There are seven *indeterminate* forms that may arise when attempting to evaluate a limit by direct substitution. These forms are

\[
\begin{array}{cccc}
0^0 & 0 \times \infty & \pm \infty & \infty - \infty \\
1^\infty & \infty^0 & 0^0 & \\
\end{array}
\]

These forms cannot be immediately determined. For instance, \(0 \times \infty\) may end up being 0, or \(\infty\), or any finite number in between. Any limit resulting in one of these forms must be handled on an individual basis.

So suppose \(\lim_{x \to a} f(x)\) results in an indeterminate form when substituting \(x = a\). For now, we will evaluate such a limit numerically by substituting other values of \(x\) that are “close” to \(a\). Later we will use some algebraic techniques to handle some \(0/0\) forms. In Calculus II, there will be other techniques, such as *L’Hospital’s Rule*, for handling the other indeterminate forms.

**Example 1.** State what type of indeterminate expression results from attempting to evaluate the limit. Then find the limit with appropriate values of \(x\). State the values of \(x\) used.

\[
\lim_{x \to -\pi} \frac{1 + \sec x}{(x + \pi)^2}
\]

**Solution.** First we substitute \(x = -\pi\):

\[
\frac{1 + \sec(-\pi)}{(-\pi + \pi)^2} = \frac{1 + (-1)}{0^2} = \frac{0}{0} ;
\]

thus, we have the indeterminate form \(0/0\). (The numerator is trying to force the expression to 0; but the denominator being near 0 is trying to make the expression infinite.)

So now we substitute values close to \(-\pi\). We must check values both a little larger and a little smaller than \(-\pi\). We verify by graphing and tracing using **Radian** mode.

Using \(x = -\pi \pm .00001\), we conclude that \(\lim_{x \to -\pi} \frac{1 + \sec x}{(x + \pi)^2} = -\frac{1}{2}\).

**Note:** After entering \(Y_1(-\pi+.001)\), we can type **2nd ENTER** to retrieve the command. Then use the cursor to edit the value and re-enter.
Example 2. State what type of indeterminate expression results from attempting to evaluate the limit. Then find the limit with appropriate values of $x$.

(a) \( \lim_{x \to 2^+} \sqrt{x-2} \tan \left( \frac{\pi x}{4} \right) \)

(b) \( \lim_{x \to 1} \frac{\ln(x^4)}{1-x} \)

(c) \( \lim_{x \to \pi^-} (\sin x)^{\tan(x/2)} \)

(d) \( \lim_{x \to \pi^-} (\tan x)^{\sin(2x)} \)

Solution. (a) For $x = 2$, $x > 2$, we have $\sqrt{x-2} = 0$. Moreover, $\frac{\pi x}{4} > \frac{\pi}{2}$ and $\tan \left( \frac{\pi x}{4} \right) = \tan \left( \frac{\pi}{2} \right) = -\infty$, the right side of the asymptote at $\pi/2$ of the tangent function. So the limit is of the form $0 \times -\infty$. That is, $\sqrt{x-2}$ is trying to force things to 0, while $\tan \left( \frac{\pi x}{4} \right)$ is trying to force things to $-\infty$. Which one wins?

Using $x$ slightly larger than 2, we see that $\lim_{x \to 2^+} \sqrt{x-2} \tan \left( \frac{\pi x}{4} \right) = -\infty$. As $x$ decreases to 2, then this function decreases without bound.

(b) For $x = 1$, we obtain $\frac{\ln 1}{1-1} = \frac{0}{0}$. Using $x = 1.00001$ and $x = 0.99999$, we see that $\lim_{x \to 1} \frac{\ln(x^4)}{1-x} = -4$.

(c) For $x = \pi$ with $x < \pi$, we have $\sin x = \sin \pi = 0$ and $\tan (x/2) = \tan (\pi/2) = +\infty$ (now we are on the left side of the asymptote of $\pi/2$ of the tangent function. So the limit is of the form $0^\infty$. So $\sin x$ is forcing things to 0 while the exponent is forcing things to $\infty$.

Using $x = \pi - 0.0001$, we see that $\lim_{x \to \pi^-} (\sin x)^{\tan(x/2)} = 0$. 
(d) Now for \( \lim_{x \to \pi/2} (\tan x)^{\sin(2x)} \), we obtain the indeterminate form \( \infty^0 \). Now \( \tan x \) is trying to force things to \( \infty \), while the exponent is trying to force things to 1.

Example 3. Not all limits can be evaluated precisely by mere function evaluation. Sometimes we obtain a non-recognizable numerical value. Consider

\[
\lim_{x \to 1^+} \frac{\ln(x) \cot(\pi x)}{\cot(\pi x)}
\]

For \( x = 1 \), we have \( \ln(1) = 0 \). For \( x = 1 \) with \( x > 1 \), then \( \pi x \approx \pi \) with \( \pi x > \pi \) so that \( \cot(\pi x) \) is approaching \( +\infty \) (i.e., it is the right side of the vertical asymptote at \( \pi \) for the cotangent graph. Thus the limit is indeterminate of the form \( 0 \times \infty \). But function evaluation does not provide an immediately recognizable limit

\[
\lim_{x \to 1^+} \frac{\ln(x) \cot(\pi x)}{\cot(\pi x)} \approx 0.3183. \text{ The actual limit is } \frac{1}{\pi}.
\]

Some Determined Forms

Some “undefined” expressions can be determined when evaluating limits. A few of these are:

1. For non-zero constant \( c \), then \( \frac{c}{0} = \pm \infty \) (i.e., there is a vertical asymptote.)

Technically, the limit does not exist since it would be infinite. However we can still say that the function increases to \( \infty \) or decreases to \( -\infty \) depending on how it approaches the asymptote.

2. \( \frac{c}{\pm \infty} = 0 \); Dividing by a very large number makes the expression approach 0.

3. \( \infty \infty = \infty \). 4. \( 0^0 = 0 \). 5. For \( 0 < c < 1 \), then \( c^\infty = 0 \). For \( c > 1 \), then \( c^\infty = \infty \).

6. For \( c > 1 \), then \( c^{-\infty} = \frac{1}{c^\infty} = \frac{1}{\infty} = 0 \). For \( 0 < c < 1 \), then \( c^{-\infty} = \frac{1}{c^\infty} = \frac{1}{0} = +\infty \).
Example 2. Explain why the limits do or do not exist:

(a) \[ \lim_{x \to \infty} e^{-x} \]

(b) \[ \lim_{x \to 0} \csc x \]

(c) \[ \lim_{x \to -1} \frac{1}{(x + 1)^2} \]

(d) \[ \lim_{x \to \infty} \frac{\sin x}{x} \]

Solutions. (a) \[ \lim_{x \to \infty} e^{-x} = \lim_{x \to \infty} \frac{1}{e^x} = \frac{1}{\infty} = 0 \]

(b) \[ \lim_{x \to 0} \csc x = \lim_{x \to 0} \frac{1}{\sin x} = \frac{1}{0} = \pm \infty, \] so the limit does not exist due to a vertical asymptote. Note that \[ \lim_{x \to 0^+} \csc x = +\infty, \] but that \[ \lim_{x \to 0^-} \csc x = -\infty. \] As \( x \) decreases to 0, \( \csc x \) increases to \( +\infty \); but as \( x \) increases to 0, \( \csc x \) decreases to \( -\infty \).

(c) \[ \lim_{x \to -1} \frac{1}{(x + 1)^2} = \frac{1}{0}, \] which means that we have no limit due to a vertical asymptote. As \( x \) approaches \( -1 \), both the numerator and denominator are positive, so the function increases to \( +\infty \) on either side of \( x = -1 \). Thus we could write \[ \lim_{x \to -1} \frac{1}{(x + 1)^2} = +\infty. \]

(d) As \( x \) increases to \( +\infty \), the function \( \sin x \) has no limit due to its oscillating, periodic behavior. But its values are always from \(-1\) to \(1\). But the denominator \( x \) keeps growing, so the fraction \( \frac{\sin x}{x} \) gets smaller and smaller. Hence, \[ \lim_{x \to \infty} \frac{\sin x}{x} = 0. \]