1. Let \( f(x) = \begin{cases} 2^x + 4 & \text{if } x < 2 \\
8 & \text{if } x = 2 \\
8 - 2 \ln(x / 2) & \text{if } x > 2 \end{cases} \).

(a) (i) \( \lim_{x \to 2^-} f(x) = \) 
(ii) \( \lim_{x \to 2^+} f(x) = \) 
(iii) \( \lim_{x \to 2} f(x) = \)

(b) Explain whether or not \( f \) is continuous at \( x = 2 \).

(c) If \( f \) is not continuous at \( x = 2 \), then explain what kind of discontinuity there is.

2. Let \( f(x) = \begin{cases} \sqrt{-x} & \text{if } x < -4 \\
2 & \text{if } x = -4 \\
(x - 4)^{1/3} & \text{if } x > -4 \end{cases} \).

(a) (i) \( \lim_{x \to -4^-} f(x) = \) 
(ii) \( \lim_{x \to -4^+} f(x) = \) 
(iii) \( \lim_{x \to -4} f(x) = \)

(b) Explain whether or not \( f \) is continuous at \( x = -4 \).

(c) If \( f \) is not continuous at \( x = -4 \), then explain what kind of discontinuity there is. Also explain if there is left-continuity or right-continuity at \( x = -4 \).
3. Let $f(x) = \begin{cases} 
-\cos(4x) & \text{if } x < \pi/3 \\
-1/2 & \text{if } x = \pi/3 \\
\sin(x/2) & \text{if } x > \pi/3 
\end{cases}$.

(a) (i) $=$ (ii) $=$ (iii) $\lim_{x \to \pi/3} f(x) =$

(b) Explain whether or not $f$ is continuous at ______.

(c) If $f$ is not continuous at ______, then explain what kind of discontinuity there is.

4. Let $f(x) = \begin{cases} 
\sqrt{x - 4} & \text{if } x \geq 8 \\
\frac{1}{x - 8} & \text{if } x < 8 
\end{cases}$.

(a) (i) $=$ (ii) $=$ (iii) $\lim_{x \to 8} f(x) =$

(b) Explain whether or not $f$ is continuous at $x = 8$.

(c) If $f$ is not continuous at $x = 8$, then explain what kind of discontinuity there is. Also explain if there is left-continuity or right-continuity at $x = 8$. 
5. Determine where the discontinuities of the function are, and state what type of discontinuities they are.

(a) \( f(x) = \frac{9 - x^2}{x^2 + 2x - 15} \)  \hspace{2cm} (b) \( f(x) = \frac{x^2 + x - 90}{100 - x^2} \)

6. Determine and describe all discontinuities.

(a) \( f(x) = \frac{9 - x^2}{x^3 - 27} \)  \hspace{2cm} (b) \( g(x) = \frac{(x + 3)(x - 5)}{(x + 3)^2 (x - 6)} \)

(c) \( k(x) = \frac{x^2 - 3x + 2}{64 - x^6} \)

(d) \( h(x) = \csc x \)  \hspace{2cm} (e) \( j(x) = \sec x \)

(f) \( g(x) = \frac{1}{\sqrt{36 - x^2}} \)  \hspace{2cm} (g) \( f(x) = \frac{10}{\sqrt{x} - 2} \)  \hspace{2cm} (h) \( h(x) = \frac{x - 9}{\sqrt{x} - 3} \)
Solutions

1. (a) (i) \( \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2^x + 4) = 8 \)  
   (ii) \( \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (8 - 2 \ln(x / 2)) = 8 \)
   (iii) Because \( \lim_{x \to 2^-} f(x) = 8 = \lim_{x \to 2^+} f(x) \), then \( \lim_{x \to 2} f(x) = 8 \) also.

(b) \( f \) is continuous at \( x = 2 \) because \( \lim_{x \to 2} f(x) = 8 = f(2) \).  (c) N/A

2. (a) (i) \( \lim_{x \to -4^-} f(x) = \lim_{x \to -4^-} \sqrt{-x} = 2 \)  
   (ii) \( \lim_{x \to -4^+} f(x) = \lim_{x \to -4^+} (x - 4)^{1/3} = -2 \)
   (iii) \( \lim_{x \to -4} f(x) \) does not exist because \( \lim_{x \to -4^-} f(x) \neq \lim_{x \to -4^+} f(x) \).

(b) \( f \) is not continuous at \( x = -4 \) because \( \lim_{x \to -4} f(x) \) does not exist.

(c) There is a jump (non-removable) discontinuity at \( x = -4 \) because and are finite, but \( \neq \). Note: Because \( \lim_{x \to -4^-} f(x) = f(-4) \), we can say that \( f \) is left continuous at \( x = -4 \).

3. (a) (i) \( \lim_{x \to \pi/3^-} f(x) = \lim_{x \to \pi/3^-} -\cos(4x) = -\cos(4\pi/3) = \frac{1}{2} \)  
   (ii) \( \lim_{x \to \pi/3^+} f(x) = \lim_{x \to \pi/3^+} \sin(x / 2) = \sin(\pi / 6) = \frac{1}{2} \)
   (iii) \( \lim_{x \to \pi/3} f(x) = \frac{1}{2} \) because \( \lim_{x \to \pi/3^-} f(x) = \frac{1}{2} = \lim_{x \to \pi/3^+} f(x) \).

(b) \( f \) is not continuous at \( x = \pi / 3 \) because \( \neq \lim_{x \to \pi/3} f(x) \).

(c) There is a “hole” in the graph (removable discontinuity) because \( \lim_{x \to \pi/3} f(x) \) exists but \( \lim_{x \to \pi/3} f(x) \neq \).
4. (a) (i) \[ \lim_{x \to 8^-} f(x) = \lim_{x \to 8^-} \frac{1}{x - 8} = -\infty \] (ii) \[ \lim_{x \to 8^+} f(x) = \lim_{x \to 8^+} \sqrt{x - 4} = 2 \]

(iii) \[ \lim_{x \to 8} f(x) \text{ does not exist because } \lim_{x \to 8^-} f(x) \text{ does not exist.} \]

(b, c) \( f \) is not continuous at \( x = 8 \) due to the vertical asymptote (infinite discontinuity) on the left side of \( x = 8 \).

Because \[ \lim_{x \to 8^+} f(x) = 2 = f(8), \] we can say that \( f \) is right continuous at \( x = 8 \).

5. (a) \[ f(x) = \frac{9 - x^2}{x^2 + 2x - 15} = \frac{(3 - x)(3 + x)}{(x + 5)(x - 3)} = \frac{-(x - 3)(3 + x)}{(x - 3)(x + 5)} \]

There is a hole in the graph at \( x = 3 \) and a vertical asymptote at \( x = -5 \).

(b) \[ f(x) = \frac{x^2 + x - 90}{100 - x^2} = \frac{(x + 10)(x - 9)}{(10 - x)(10 + x)} = \frac{(x + 10)(x - 9)}{(x + 10)(10 - x)} \]

There is a hole in the graph at \( x = -10 \) and a vertical asymptote at \( x = 10 \).

6. (a) \[ f(x) = \frac{9 - x^2}{x^3 - 27} = \frac{(3 - x)(3 + x)}{(x - 3)(x^2 + 3x + 9)} = \frac{-(3 + x)}{x^2 + 3x + 9} \text{ for } x \neq 3. \]

The only root of the denominator is \( x = 3 \) (from solving \( x^3 - 27 = 0 \)), but \( x = 3 \) is also a root of the numerator of equal multiplicity. So there is a hole in the graph (removable discontinuity) at \( x = 3 \).

(b) \[ g(x) = \frac{(x + 3)(x - 5)}{(x + 3)^2(x - 6)} = \frac{(x - 5)}{(x + 3)(x - 6)} \text{ for } x \neq -3. \] The denominator roots of \( x = -3 \) and \( x = 6 \) both create vertical asymptotes (infinite discontinuities).
(c) \[ k(x) = \frac{x^2 - 3x + 2}{64 - x^6} = \frac{(x-1)(x-2)}{(8-x^3)(8+x^3)} = \frac{(x-1)(x-2)}{(2-x)(4+2x+x^2)(2+x)(4-2x+x^2)} = \frac{-(x-1)}{(4+2x+x^2)(2+x)(4-2x+x^2)} \text{ for } x \neq 2. \]

From solving \(64 - x^6 = 0\), we obtain \(x = \pm \sqrt[6]{64} = \pm 2\) as the only roots of the denominator. The denominator root of \(x = 2\) is also a root of the numerator of equal multiplicity. So there is a removable discontinuity (hole in the graph) at \(x = 2\). But the denominator root of \(x = -2\) creates a vertical asymptote (infinite discontinuity).

(d) \(h(x) = \csc x\) has infinite discontinuities (vertical asymptotes) whenever \(\sin x = 0\), which occur at \(x = k\pi\), where \(k\) is any integer.

(e) \(j(x) = \sec x\) has infinite discontinuities (vertical asymptotes) whenever \(\cos x = 0\), which occur at \(x = \frac{\pi}{2} + k\pi\), where \(k\) is any integer.

(f) \(g(x) = \frac{1}{\sqrt{36 - x^2}}\) has infinite discontinuities (vertical asymptotes) whenever \(36 - x^2 = 0\), which occur at \(x = \pm 6\).

(g) \(f(x) = \frac{10}{\sqrt{x-2}}\) has a vertical asymptote when \(\sqrt{x} - 2 = 0\), which is at \(x = 4\).

(h) \(k(x) = \frac{x-9}{\sqrt{x-3} = \frac{x-9}{\sqrt{x-3} \cdot \sqrt{x+3}} = \frac{(x-9)(\sqrt{x+3})}{x-9} = \sqrt{x+3} \text{ for } x \neq 9. \) There is a hole in the graph (removable discontinuity) at \(x = 9\).