Annuities are a popular and effective method of saving for retirement. In a sense, they are the opposite of a mortgage. First, an initial deposit of \( P \) is made. At the end of the month and at the end of each month thereafter, a monthly deposit of \( M \) is made. Each month, you increase the principle upon which you are drawing interest, unlike a mortgage on which each month you decrease the principle upon which you are paying interest.

Assuming a final deposit is made at the end of the last month, then the future value \( FV \) in \( t \) years is given by the formula:

\[
FV = P \left(1 + \frac{r}{12}\right)^{12t} + \frac{12M}{r} \left[ \left(1 + \frac{r}{12}\right)^{12t} - 1 \right]
\]

**Note:** The future value is the annuity’s worth after \( t \) years. The *present value* is the amount that should be deposited now to give the same future value without having to make monthly payments.

**Example 1.** Suppose you start an annuity by depositing $30,000 and adding $500 a month for the next 30 years. The annuity pays 6.6% interest.

(a) What is the future value in 30 years and how much is actually paid in?
(b) What is the present value of this annuity?
(c) If you wanted $1,000,000 after 30 years but still only added $500 a month, then what should the initial deposit have been?
(d) If you wanted $1,000,000 after these 30 years and still deposited only $30,000, then how much should the monthly payment have been?

**Solution.**

(a) The future value in 30 years is given by

\[
FV = 30000 \times \left(1 + \frac{0.066}{12}\right)^{12 \times 30} + \frac{12 \times 500}{0.066} \left[ \left(1 + \frac{0.066}{12}\right)^{12 \times 30} - 1 \right]
\]

\[
= 30000 \times (1.0055)^{360} + \frac{6000}{0.066} [(1.0055)^{360} - 1]
\]

\[
= 216,106.70 + 563,959.70
\]

\[= \$780,066.40.\]

The amount paid in is 30,000 + 30 \times 12 \times 500 = $210,000. So you earn over $570,000 in interest over the course of 30 years.

(b) The present value is the required deposit to attain $780,066.40 in 30 years with no monthly payments. So solve for \( P \) in the equation \( P \times (1.0055)^{360} = 780,066.40 \rightarrow P = 780,066.40 \div (1.0055)^{360} = \$108,289.07.\)
(c) Solve for \( P \) in the equation \( P \times (1.0055)^{360} + 563,959.70 = 1,000,000 \)

\[
\rightarrow P \times (1.0055)^{360} = 436,040.30 \rightarrow P = 436,040.30 \div (1.0055)^{360} = \$60,531.25.
\]

(d) Solve for \( M \) in the equation \( 216,106.70 + \frac{12 \times M}{0.066} [(1.0055)^{360} - 1] = 1,000,000 \)

\[
\rightarrow \frac{12 \times M}{0.066} [(1.0055)^{360} - 1] = 783893.30
\]

\[
\rightarrow M = 783893.30 \times 0.066 \div 12 \div [(1.0055)^{360} - 1] \rightarrow M = \$694.99.
\]

**Simple Interest Annuities**

For annuities that pay simple interest, the additional payments are added just at the end of the year. In this case, the future value \( FV \) in \( t \) years is given by the formula:

\[
FV = P(1 + r)^t + \frac{M}{r} [(1 + r)^t - 1]
\]

**Example 2.** You begin a Roth IRA with a deposit of $3000. At the end of every year thereafter, you add another $3000. The IRA guarantees a minimum 4.5% return each year. What is the minimum future value in 25 years? How much is paid in over 25 years? What is the present value?

**Solution.** The future value is given by

\[
FV = P(1 + r)^t + \frac{M}{r} [(1 + r)^t - 1] = 3000 \times (1.045)^{25} + \frac{3000}{0.045} [(1.045)^{25} - 1] = \$142,711.93.
\]

To attain this amount, you have only paid in $3000 + 25 \times 3000 = \$78,000.

**Note:** The interest gained on such a Roth IRA is tax-free! However the contributions made each year are not tax-deductible.

To find the present value, solve for \( P \) in the equation: \( P \times (1.045)^{25} = 142,711.93 \rightarrow P = 142,711.93 \div (1.045)^{25} = \$47,484.63 \). In other words, if you invest \$47,484.63 with simple yearly interest of 4.5%, then you will attain $142,711.93 in 25 years without any additional yearly payments.

**Exercise.** You start an annuity by depositing $5000 and adding $600 a month for the next 25 years. The annuity pays 6% interest.

(a) What is the future value in 25 years and how much is actually paid in?
(b) Solve for the present value of this annuity.
(c) Suppose you want $500,000 after 25 years but you will only add $600 a month. Solve for what the initial deposit should be.
(d) Suppose you want $500,000 after these 25 years and you only deposit $5000. Solve for what the monthly payment should be.
Solution

(a) The future value in 25 years is

\[
FV = 5000 \times \left(1 + \frac{0.06}{12}\right)^{12 \times 25} + \frac{12 \times 200}{0.06} \left(1 + \frac{0.06}{12}\right)^{12 \times 25} - 1
\]

\[
= 5000 \times (1.005)^{300} + 40,000 \times (1.005)^{300} - 1
\]

\[
= 22,324.85 + 138,598.79
\]

\[
= \$160,923.64.
\]

The amount paid in is \(5000 + 25 \times 12 \times 200 = \$65,000\).

(b) The present value is the initial amount \(P\) that could be deposited with no monthly payments in order to achieve the same future value.

\[
P \times (1.005)^{300} = 160,923.64 \quad \rightarrow \quad P = \frac{160,923.64}{(1.005)^{300}} = \$36,041.37.
\]

(c) Solve for \(P\) in the equation \(P \times (1.005)^{300} + 138,598.79 = 200,000\)

\[
\rightarrow \quad P \times (1.005)^{300} = 61,401.21
\]

\[
\rightarrow \quad P = \frac{61,401.21}{(1.005)^{300}} \quad \rightarrow \quad P = \$13,751.76.
\]

(d) Solve for \(M\) in the equation \(22,234.85 + \frac{12M}{.06}[(1.005)^{300} - 1] = 200,000\).

\[
\rightarrow \quad \frac{12M}{.06}[(1.005)^{300} - 1] = 177,675.15
\]

\[
\rightarrow \quad M = 177,675.15 \times 0.06 \div 12 \div [(1.005)^{300} - 1] \quad \rightarrow \quad M = \$256.39.
\]