Let $p$ be the probability of “success” on each and every attempt, and let $q$ be the probability of failure, where $q = 1 - p$. For example, if $p = 0.15$, then $q = 0.85$. We wish to find probabilities involving a sequence of independent attempts. Here independent means that the outcome of one attempt does not affect and is not affected by the outcome of any other attempt. It applies in particular when we are sampling with replacement.

**Example 1.** Draw 6 cards in a row each time replacing the card and reshuffling so that you are always drawing from a full deck of 52. What is the probability of drawing the sequence Non-Heart, Heart, Non-Heart, Non-Heart, Non-Heart, Heart?

**Solution.** Because we are always replacing the card and reshuffling, the probability of a Heart is always $p = \frac{1}{4}$, and the probability of a non-Heart is always $q = \frac{3}{4}$. So the probability of drawing the desired sequence is

$$P(NH, H, NH, NH, NH, H) = \frac{3}{4} \times \frac{1}{4} \times \frac{3}{4} \times \frac{3}{4} \times \frac{1}{4} = \frac{81}{4096} = 0.019775.$$  

**Formulas for a Sequence of Independent Attempts**

Again, let $p$ be the probability of success and let $q = 1 - p$ be the probability of failure. Assume we make a sequence of independent attempts.

- The average number of tries needed to succeed is $\frac{1}{p}$.

  Given just $n$ attempts,

- The probability of all successes is $p^n$.

- The probability of all failures is $q^n$.

If you don’t have all failures, then you have at least one success.

- So the probability of at least one success is $1 - q^n$.

- The average number of successes is $np$.

- The probability of exactly $k$ successes in $n$ attempts is

$$C(n, k) \times p^k \times q^{n-k}.$$
**Example 2.** Suppose you have a 12% chance of winning at a slot machine.

(a) What is the average number of tries needed to win?
(b) In 14 attempts, what is the chance of all wins? all losses? at least one win?
(c) What is the average number of wins in 14 tries?
(d) What is the probability of exactly 2 wins in 14 tries?

*Solution.* (a) On average \( \frac{1}{p} = \frac{1}{0.12} \approx 8.33 \) tries are needed to win.

(b) In 14 tries, the probability of all wins is \( p^{14} = 0.12^{14} \approx 1.284 \times 10^{-13} \approx 0 \). The probability of all losses is \( q^{14} = 0.88^{14} \approx 0.167 \). Thus, the probability of at least one win in 14 tries is \( 1 - q^{14} = 1 - 0.167 \approx 0.833 \).

(c) The average number of wins in 14 tries is \( n p = 14 \times 0.12 = 1.68 \).

(d) The probability of exactly 2 wins in 14 tries is

\[
C(14, 2) \times 0.12^2 \times 0.88^{12} = 91 \times 0.12^2 \times 0.88^{12} \approx 0.2826.
\]

**The Law of Try Try Again**

The chance of all failures in \( n \) attempts is \( q^n \); thus, the chance of at least one success in \( n \) attempts is \( 1 - q^n \). As \( n \) increases, the chance of at least one success gets higher and higher.

How many times \( n \) should we attempt so that the chance of at least one success is as high as \( r \), where \( r \) is usually 0.90, 0.95 or 0.99? We must find the number of tries \( n \) needed to make \( 1 - q^n \geq r \). We can do so by trial and error, or with the formula below:

To make the chance of at least one success as high as \( r \), we should try at least

\[
\frac{\log(1 - r)}{\log(1 - p)} = \frac{\log(1 - r)}{\log(q)} \text{ times}
\]

(rounded up to the next integer).
Example 3. Suppose you have a 12% chance of winning at a slot machine. How many times should you play so that the chance of at least one win is 0.95?

Solution. Here \( r = 0.95 \) and \( \frac{\log(1 - r)}{\log(q)} = \frac{\log(0.05)}{\log(0.88)} \approx 23.43 \); so we need \( 24 \) attempts.

Note: The actual probability of at least one success in 24 tries is \( 1 - 0.88^{24} \approx 0.9535 \).

Example 4. Your chance of winning at craps is about 0.49.

(a) What is the average number of tries needed to win?

Now suppose you play 10 times. What is the probability of getting

(b) all wins

(c) no wins

(d) at least one win

(e) exactly 5 wins?

(f) What is the average number of wins in 10 tries?

(g) To make the chance of winning within \( n \) plays as high as 0.90, how many plays \( n \) are needed?

Answers. (a) \( \frac{1}{0.49} \approx 2.04 \) tries

(b) \( 0.49^{10} \approx 0.000798 \)

(c) \( 0.51^{10} \approx 0.00119 \)

(d) \( 1 - 0.51^{10} \approx 1 - 0.00119 = 0.99881 \) is the chance of at least one win in 10 tries.

(e) \( C(10, 5) \times 0.49^5 \times 0.51^5 = 252 \times 0.49^5 \times 0.51^5 \approx 0.2456 \).

(e) The average number of wins in 10 tries is \( 10 \times 0.49 = 4.9 \).

(f) \( \frac{\log(1 - r)}{\log(q)} = \frac{\log(0.10)}{\log(0.51)} \approx 3.42 \rightarrow \) only 4 tries are needed.
Exercise

You are applying to professional school and due to enormous competition, you only have around a 16% chance of being accepted to any particular school.

(a) What is the average number of applications you need to get an acceptance?

If you apply to 6 schools, what is your probability of

(b) being accepted to all?  
(c) being accepted to none?

(d) being accepted to at least one?  
(e) being accepted to exactly 2 schools?

(e) What is the average number of acceptances in 6 applications?

(g) How many applications \( n \) are needed to make the probability of at least one acceptance as high as 0.96?