We now shall consider some probabilities that result when sampling *without replacement* either in order or without regard to order. The process of drawing cards illustrates the ordered sampling.

**Example 1.** The cards 10 through Ace are considered to be "High" cards. When drawing 5 cards in sequence *without replacement*, what is the probability of

(a) a High card, then a Low card, then a Low card, then a High card, then a High card?

(b) a Face card, then a Number card, then a Number card, then a Number card, then an Ace?

(c) a Heart, then a Diamond, then a Diamond, then a club, then a Heart?

**Solution.** (a) To start, there are 20 high cards and 32 low cards out of 52. By not replacing the cards before drawing the next, the numbers decrease by one as the next cards are drawn. Thus, the desired probability is

\[
\frac{20}{52} \times \frac{32}{51} \times \frac{31}{50} \times \frac{19}{49} \times \frac{18}{48} \approx 0.0217564.
\]

(b) To start, there are 12 face cards and 36 number cards out of 52. So the desired probability is

\[
\frac{12}{52} \times \frac{36}{51} \times \frac{35}{50} \times \frac{34}{49} \times \frac{4}{48} \approx 0.0065934.
\]

(c) To start, there are 13 in each suit. Now the desired probability is

\[
\frac{13}{52} \times \frac{13}{51} \times \frac{12}{50} \times \frac{13}{49} \times \frac{12}{48} \approx 0.001.
\]

**Sampling Without Regard to Order**

Now suppose we have a population of \( N \) elements that are divided into two types: Type I which has \( A \) elements, and Type II which has \( B \) elements, where \( A + B = N \). For example, a standard deck of \( N = 52 \) playing cards can be divided in many ways. Type I could be “Hearts” and Type II could be “All Others.” Then there are \( A = 13 \) Hearts and \( B = 39 \) Others.
Total is \( A + B = N \) (52 cards)

Now suppose a random sample of size \( n \) is taken all at once from the entire population of \( N \) objects. We then are sampling without replacement and without regard to order. We wish to find the probability of having exactly \( k \) elements of Type I in this sample.

When sampling all at once without replacement, the probability of having exactly \( k \) elements of Type I is given by

\[
\frac{C(A, k) \times C(B, n - k)}{C(N, n)}
\]

The denominator comes from the total number of ways to choose \( n \) from the entire group of \( N \). The numerator comes from choosing \( k \) out of \( A \) from Type I, then choosing the rest from \( B \) of Type II.

The average number of objects of Type A in the sample of size \( n \) is given by

\[
n \times \frac{A}{N} = n p,
\]

where \( p = \frac{A}{N} \) is the prob. of Type A.
Example 2. Suppose a hand of 5 cards is dealt from a deck of 52.

(a) What are the possibilities for the number of Hearts in the hand?
(b) What is the probability of there being exactly 2 Hearts in the hand?
(c) What is the average number of Hearts that will be in the hand?
(d) What is the average number of non-Hearts that will be in the hand?

Solution. There are $N = 52$ cards, $A = 13$ Hearts, and $B = 39$ others. We take a sample of size $n = 5$. (a) We can have anywhere from 0 to 5 Hearts in the hand.

(b) The probability of exactly 2 Hearts is

$$\frac{C(13, 2) \times C(39, 3)}{C(52, 5)} = \frac{78 \times 9139}{2,598,960} = \frac{712,842}{2,598,960} \approx 0.27428.$$

The denominator comes from the total number of ways to choose 5 cards from 52. The numerator comes from choosing 2 out of 13 Hearts, then choosing 3 from the other 39 cards.

(c) The average number of Hearts in a 5 card deal is $n \times \frac{A}{N} = 5 \times \frac{13}{52} = 1.25$.

(d) The average number of non-Hearts in a 5 card deal must be $5 - 1.25 = 3.75$.

Example 3. At a campus protest rally, there are 18 students and 12 faculty. The cops choose 20 people at random to bust for disorderly conduct. In this group of 20,

(a) What are the possibilities for the number of students busted? What are the possibilities for the number of faculty busted?
(b) What is the probability of there being exactly 12 students busted?
(c) What is the average number of students chosen in a sample of size 20?
(d) What is the average number of faculty chosen in a sample of size 20?

Solution. There are $N = 30$ people, $A = 18$ students, and $B = 12$ faculty. We take a sample of size $n = 20$.

(a) Since 20 are chosen and there are only 12 faculty, then at least 8 students must be chosen. So anywhere from 8 to 18 students may be busted. If there are 8 students, then there must be 12 faculty. If there are 18 students, then there must be only 2 faculty. So anywhere from 2 to 12 faculty would have to be busted.

(b) The probability of exactly 12 students being busted is

$$\frac{C(18, 12) \times C(12, 8)}{C(30, 20)} = \frac{18,564 \times 495}{30,045,015} = \frac{9,189,180}{30,045,015} \approx 0.305847.$$

The denominator comes from the total number of ways to choose 20 people from 30. The numerator comes from choosing 12 out of 18 students, then choosing 8 from 12 faculty.
(c, d) The average number of students in a sample of 20 is \(20 \times \frac{18}{30} = 12\). And so the average number of faculty in a sample of 20 is 8.

**Example 4.** There are 80 people serving jury duty. One of these people is you. Of these 80 people, 14 are chosen at random to try a particular case. What is the probability that you are chosen for the case?

**Solution.** Consider yourself the 1 person of Type A. Then there are 79 of Type B. The probability of exactly 1 of Type A being chosen (i.e., you) is

\[
\frac{C(1,1) \times C(79, 13)}{C(80, 14)}
\]

Since \(C(1, 1) = 1\), this value simplifies as follows:

\[
\frac{C(79, 13)}{C(80, 14)} = \frac{79!}{13! \times 66!} = \frac{79!}{13!} \times \frac{14!}{80!} = \frac{14}{80}.
\]

So the chance of you being picked is 14 / 80 since there are 80 in the group and 14 are chosen.
Exercises

1. When drawing 6 cards in order from a standard deck, what is the probability of drawing
   - a Low card, then a High card, then a High card,
   - then a Low card, then a Low card, then a Low card,

(a) if drawing without replacement of cards?
(b) if drawing with replacement and re-shuffling of cards?

2. Suppose a hand of 7 cards is dealt.

(a) What are the possibilities for the number of Aces in the hand?
(b) What is the probability of there being exactly 1 Ace in the hand?
(c) What is the average number of Aces in a 7 card deal?
(d) What is the average number of non-Aces in a 7 card deal?

3. In a drama class, there are 12 females and 10 males. A group of 15 is to be chosen at random to read a screenplay.

(a) What are the possibilities for the number of males chosen? For the number of females?
(b) What is the probability of there being exactly 6 males chosen?
(c) What is the average number of males chosen in a sample of 15? What is the average number of females chosen?

4. In another class, there are 10 males and 17 females. A random group of 14 is chosen.

(a) What are the possibilities for the number of females chosen? For the number of males?
(b) What is the probability of there being exactly 9 females chosen?
(c) What is the average number of females chosen in a sample of 14? What is the average number of males chosen?

5. You and your friend are in a police line-up of 20 people. A “witness” claims he can identify the 6 people who held up the bank. If the witness just fingers 6 people at random from the group of 20, what is the probability that both you and your friend are picked? (Simplify the factorials in fractional form.)
Solutions

1. (a) \[ \frac{32}{52} \times \frac{20}{51} \times \frac{19}{50} \times \frac{31}{49} \times \frac{30}{48} \times \frac{29}{47} \approx 0.0223736 \]

(b) \[ \frac{32}{52} \times \frac{20}{52} \times \frac{20}{52} \times \frac{32}{52} \times \frac{32}{52} \times \frac{32}{52} \approx 0.021215 \]

2. There are \( N = 52 \) cards, \( A = 4 \) Aces, and \( B = 48 \) others. The sample size is \( n = 7 \).

(a) We can have anywhere from 0 to 4 Aces in the hand.

(b) The probability of exactly 1 Ace is

\[ \frac{C(4,1) \times C(48,6)}{C(52,7)} = \frac{4 \times 12,271,512}{133,784,560} = \frac{49,086,048}{133,784,560} \approx 0.3669. \]

The denominator comes from the total number of ways to choose 7 cards from 52. The numerator comes from choosing 1 out of 4 Aces, then choosing 6 from the other 48 cards.

(c, d) The average number of Aces in a 7 card deal is \( 7 \times \frac{4}{52} = \frac{7}{13} \approx 0.53846 \), and the average number of non-Aces is \( 7 - \frac{7}{13} \approx 6.46154 \).

3. There are \( N = 22 \) students, \( A = 10 \) males, and \( B = 12 \) females. We take a sample of size \( n = 15 \).

(a) Since there are only 12 females, we must have at least 3 males in the sample. But we can have at most 10 males in the sample. So the number of males must be from 3 to 10.

If there are 3 males, then we must have 12 females. If there are 10 males, then we must have 5 females. So the number of females must be from 5 to 12.

(b) The probability of exactly 6 males being chosen is

\[ \frac{C(10,6) \times C(12,9)}{C(22,15)} = \frac{210 \times 220}{170,544} = \frac{46,200}{170,544} \approx 0.2709. \]

The denominator comes from the total number of ways to choose 15 students from 22. The numerator comes from choosing 6 out of 10 males, then choosing 9 from the 15 females.

(c) The average number of males in a sample of 15 is \( 15 \times \frac{10}{22} \approx 6.818 \), and the average number of females is \( 15 \times \frac{12}{22} \approx 8.182 \).
There are $N = 27$ students, $A = 17$ females, and $B = 10$ males. We take a sample of size $n = 14$.

(a) Since there are only 10 males, we must have at least 4 females in the sample. The number of females must be from 4 to 14. And the number of males must be from 0 to 10.

(b) The probability of exactly 9 females being chosen is
\[
\frac{\binom{17}{9} \times \binom{10}{5}}{\binom{27}{14}} = \frac{24,310 \times 252}{20,058,300} = \frac{6,126,120}{20,058,300} \approx 0.3054.
\]

The denominator comes from the total number of ways to choose 14 students from 27. The numerator comes from choosing 9 out of 17 females, then choosing 5 from the 10 males.

(c) The average number of females in a sample of 14 is $14 \times \frac{17}{27} \approx 8.8148$, and the average number of males is $14 \times \frac{10}{27} \approx 5.1852$.

You and your friend are the 2 persons of Type $A$. Then there are 18 of Type $B$. The probability of exactly 2 of Type $A$ being chosen (i.e., both you and your friend) is
\[
\frac{\binom{2}{2} \times \binom{18}{4}}{\binom{20}{6}}.
\]

Since $\binom{2}{2} = 1$, this value simplifies as follows:
\[
\frac{\binom{18}{4}}{\binom{20}{6}} = \frac{18!}{14! \times 4!} \times \frac{14! \times 4!}{20!} = \frac{18!}{20!} \times \frac{6!}{20 \times 19} = \frac{6 \times 5}{20} \times \frac{5}{19}
\]

In other words, the chance of you being picked is $\frac{6}{20}$, then the chance of your friend being picked is $\frac{5}{19}$. The chance of both of you being picked is $\frac{6}{20} \times \frac{5}{19} = \frac{3}{38} \approx 0.078947$. 