Combinations

Given a set of $N$ distinct objects, we often want to choose a smaller group of these objects, all at once, without repeats, and without regard to order. Such a choice is called a combination or a selection. The resulting smaller group is a subset of the original set.

For example, when playing the Pick–6 Lotto, we must choose six numbers from the group 1–42. Obviously, we cannot repeat numbers. But more importantly, the order does not matter. In other words, we are not choosing a sequence like a serial number or password where we have a first number, then a second number, then a third, etc. We simply bubble in any six numbers on the card and if they come up in any order, then we win. We simply need a subset of size 6 from the group of 42.

So we might bubble in 9, 18, 27, 30, 33, 38. If the numbers come up in the drawing as 18–38–33–9–30–27, then we would win because the order does not matter.

(Note that with a permutation, like in a serial number, the order does matter so that 18–38–33–9–30–27 would be different from 38–18–9–33–27–30.)

The Formula for $C(N, k)$

“$N$ Choose $k$”

The expression $C(N, k)$, also written $nCr$ on some calculators, is computed by

$$C(N, k) = \frac{N!}{k!(N-k)!}.$$

This value gives the number of combinations (or subsets) of size $k$ from a group of $N$. For example, $C(10, 6) = \frac{10!}{6!(10-6)!} = \frac{10!}{6!4!}$. It is best to use the $nCr$ button to compute this value: 10 $nCr$ 6 = 210. So there are 210 subsets of size 6 from a group of 10.

If you are going to compute it directly by dividing the factorials, then you must be careful to divide correctly in one of two ways:

$$10! ÷ 6! ÷ 4! = 210 \quad \text{or} \quad 10! ÷ (6! × 4!) = 210.$$

Example 1. In the Pick–6 Lotto, there are $C(42, 6) = 5,245,786$ ways to select numbers. The probability of winning with one play is $p = \frac{1}{5,245,786}$ (less than 1 in 5 million).
Exercise

1. Use your calculator to compute the following:

   (a) $C(12, 8)$  
   (b) $C(52, 7)$  
   (c) $C(45, 6)$

Combinations from a Set

Now given a set of $N$ distinct objects, if we want to choose $k$ of these all at once without repeats and without regard to order, then we are creating a combination or selection. The number of such combinations is given by $C(N, k)$ which is also the number of possible subsets of size $k$ from the original set of $n$ objects.

**Example 2.** Consider the set of six letters $\{a, b, c, d, e, f\}$. How many ways are there to

   (a) Align two of these in order with possible repeats? 
   (b) Align two of these in order with no repeats? 
   (c) Choose two of these? (By saying “choose” or “select” it is understood that there are no repeats and order doesn’t matter.)
   (d) How many subsets of size two are there? What are they?

**Solution.** In general, we start with $N = 6$ objects and we somehow need $k = 2$ of these.

   (a) In order, repeats allowed: General formula is $N^k$, here there are $6^2 = 6 \times 6 = 36$ possibilities: $aa \ a \ a \ b \ a \ c \ \ a \ d \ \ a \ e \ \ a \ f \ \ b \ a \ b \ b \ b \ b \ a \ c \ \ etc$. Note: $ab \neq ba$.

   (b) In order, no repeats (i.e., a permutation): General formula is $P(N, k)$, here there are $P(6, 2) = 30$ possibilities. (Because $k$ is small in this case, it is easier to multiply out $6 \times 5 = 30$.) Out of the 36 possibilities in Part (a), simply omit the 6 cases with repeats: $aa \ b \ b \ c \ c \ c \ etc$. which leaves the other 30 possibilities.

   (c) Choose $k$: General formula is $C(N, k)$, here there are $C(6, 2) = 15$ combinations.

   (d) Subsets of size $k$: Again the formula is $C(N, k)$; so there are $C(6, 2) = 15$ subsets. These subsets are $\{a, b\}, \{a, c\}, \{a, d\}, \{a, e\}, \{a, f\}, \{b, c\}, \{b, d\}, \{b, e\}, \{b, f\}, \{c, d\}, \{c, e\}, \{c, f\}, \{d, e\}, \{d, f\}, \{e, f\}$. Note: In this case, $\{a, b\}$ is the same as $\{b, a\}$, so we do not list them both.

**Example 3.** How many possible 5-card poker hands are there?

**Solution.** Simply deal 5 cards from 52. Cards cannot be repeated, and the order of the cards in your hand does not matter. Thus, a 5-card hand is a combination. There are $C(52, 5) = 2,598,960$ possible hands.
Exercises

2. Consider the four elements \{\alpha, \beta, \mu, \sigma\}. How many ways are there to

(a) Align two of these in order with possible repeats? List all possibilities.
(b) Align two of these in order with no repeats? List all possibilities.
(c) Choose two of these? List all possible subsets of size 2.

3. With your pizza coupon, you get 3 toppings chosen from 12 possible (black olive, pepperoni, sausage, mushroom, etc.) How many possible choices do you have to create a three-topping pizza if you must have three different kinds of toppings?

4. Consider a group of 15 people. How many ways are there to

(a) Choose 3? Choose 12?  
(b) Choose 4? Choose 11?  
(c) Choose 5? Choose 10?  
(d) Choose 6? Choose 9?  
(e) Why do you obtain the same answer for both questions in each part?
(f) From a group of 30, what would choosing 10 be the same as?
(g) From a group of 40, what would choosing 25 be the same as?

5. In the Pick–5 Lotto, the player chooses 5 numbers from 1–35.  
(a) What is the probability of winning with one play?  
(b) What is the probability of winning if you buy 8 plays all with different choices?

6. In Powerball, you must match 5 numbers from 59 (in any order), then match the powerball from 35 choices. What is your probability of winning with one play?

7. A poker flush is a hand with 5 cards in the same suit.  (which includes the straight flushes). How many poker flushes are there in the suit of diamonds?
Answers

1. (a) 495         (b) 133,784,560          (c) 8,145,060

2. (a) 16: $\alpha \alpha, \alpha \beta, \alpha \mu, \alpha \sigma, \beta \alpha, \beta \beta, \ldots$  (b) $4 \times 3 = 12$: all those from Part (a) except $\alpha \alpha, \beta \beta, \mu \mu, \sigma \sigma$,  (c) $C(4, 2) = 6$: $\alpha \beta, \alpha \mu, \alpha \sigma, \beta \mu, \beta \sigma, \mu \sigma$

3. To choose 3 from 12, there are $C(12, 3) = 220$ possibilities.

4. Consider a group of 15 people. How many ways are there to

(a) $C(15, 3) = 455 = C(15, 12)$         (b) $C(15, 4) = 1365 = C(15, 11)$
(c) $C(15, 5) = 3003 = C(15, 10)$         (d) $C(15, 6) = 5005 = C(15, 9)$

(e) Choosing $k$ to take from $N$ is the same as choosing $N-k$ to leave behind. So $C(N, k) = C(N, N-k)$.

(f) $C(30, 10) = C(30, 20)$
(g) $C(40, 25) = C(40, 15)$

5. (a) There are $C(35, 5) = 324,632$ ways to choose; $p = \frac{1}{324,632}$.

(b) With 8 plays, the chance of winning is $\frac{8}{324,632} = \frac{1}{40,579}$.

6. There are $C(59, 5) \times 35 = 175,223,510$ ways to pick in the Powerball lottery. Thus your chance of winning with one play is $\frac{1}{175,223,510}$ (less than 1 in 175 million).

7. There are $C(13, 5) = 1287$ possible poker flushes in the suit of diamonds.