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T_i-ordered reflections

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Abstract. We present a construction which shows that the T_i-ordered reflection ($i \in \{0, 1, 2\}$) of a partially ordered topological space $(X, \tau, \leq)$ exists and is an ordered quotient of $(X, \tau, \leq)$. We give an explicit construction of the T_0-ordered reflection of an ordered topological space $(X, \tau, \leq)$, and characterize ordered topological spaces whose T_0-ordered reflection is T_1-ordered.

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1. Introduction

The T_0-, T_1-, and T_2-reflections of a topological space have long been of interest to categorical topologists. Methods of constructions of these in the category TOP are described in the references given in [3] (see p. 302). Here, we consider the corresponding concepts of T_i-ordered reflections in the category ORDTOP of partially ordered topological spaces with continuous increasing functions as morphisms. In Section 2, we construct the T_i-ordered reflection ($i = 0, 1, 2, S2$) of a partially ordered topological space $(X, \tau, \leq)$. The construction is extrinsic, occurring in the category PREORDTOP of preordered topological spaces with continuous increasing functions as morphisms, which contains ORDTOP as a subcategory. In Section 3, we give an intrinsic construction of the T_0-ordered reflection of a partially ordered space $(X, \tau, \leq)$ and examine some properties of this reflection.

A preordered topological space $(X, \tau, \preceq)$ is a set X with a topology τ and a preorder $\preceq$. Following the notation of Nachbin ([7]), for $A \subseteq X$, the increasing hull of A is $i(A) = \{y \in X : \exists a \in A \text{ with } a \preceq y\}$. A set A is an *increasing set*

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if $A = i(A)$. The closed increasing hull $I(A)$ of $A \subseteq X$ is the smallest closed increasing set containing A. Decreasing sets, decreasing hulls $d(A)$, and closed decreasing hulls $D(A)$ are defined dually. A set is *monotone* if it is increasing or decreasing. There are many compatibility conditions between the topology and order of a preordered topological space which one may stipulate. These include the convex topology condition (τ has a subbase of monotone open sets) or the ordered separation axioms, some of which are defined below. Our preordered spaces need not satisfy any of these compatibility conditions.

Suppose $(X, \tau, \preceq)$ is a preordered topological space. It is well-known that the preorder $\preceq$ induces an equivalence relation $\sim = G(\preceq) \cap G(\preceq^{-1})$ on X defined by $x \sim y$ if and only if $x \preceq y$ and $y \preceq x$. The preordered topological space $(X, \tau, \preceq)$ is said to be T_0-*preordered* if any of the following equivalent statements holds.

(a) $x \not\sim y \Rightarrow [I(x) \neq I(y)$ OR $D(x) \neq D(y)]$
(b) If $I(x) = I(y)$ and $D(x) = D(y)$, then $x \sim y$.
(c) If $x \not\sim y$, there exist a monotone open neighborhood of one of the points which does not contain the other point.

Observe that if $\preceq$ is a partial order, then the relation $\sim$ is equality.

A preordered topological space $(X, \tau, \preceq)$ is T_1-*preordered* if $i(x)$ and $d(x)$ are closed for every $x \in X$, or equivalently, if $x \not\preceq y$ in X implies there exists an open increasing neighborhood of x which does not contain y and there exists an open decreasing neighborhood of y which does not contain x.

A preordered topological space $(X, \tau, \preceq)$ is T_2-*preordered* if there is an increasing neighborhood of x disjoint from some decreasing neighborhood of y whenever $x \not\preceq y$. Equivalently, $(X, \tau, \preceq)$ is T_2-preordered if the preorder $\preceq$ is closed in $(X, \tau) \times (X, \tau)$.

A preordered topological space $(X, \tau, \preceq)$ is *strongly* T_2-*preordered*, or for notational convenience, T_{S2}-*preordered*, if there is an increasing *open* neighborhood of x disjoint from some decreasing *open* neighborhood of y whenever $x \not\preceq y$.

If $\preceq$ is a partial order, then $(X, \tau, \preceq)$ is a partially ordered topological space, or simply an ordered topological space. If the preorder of a T_i-preordered topological space $(X, \tau, \leq)$ is a partial order, we will say $(X, \tau, \leq)$ is T_i-ordered. We will typically denote preorders by $\preceq$ and partial orders by $\leq$. To avoid confusion when indicating inclusions, we may represent a preorder $\sqsubseteq$ by its graph $G(\sqsubseteq)$.

2. Existence of T_i-ordered reflections

A special quotient. The definition of a T_0-preordered topological space $(X, \tau, \preceq)$ involved the equivalence relation $\sim$ on X defined by $a \sim b$ if and only if $a \preceq b$ and $b \preceq a$. For any preordered topological space $(X, \tau, \preceq)$, we obtain a partially ordered topological space by giving $X/\sim$ the quotient topology $\tau/\sim$ and the order $\leq$ defined by $[a] \leq [b]$ if and only if $a \preceq b$. The following properties of this quotient construction are easily verified.

Proposition 2.1. *Suppose $\preceq$ is a preorder on a set X, $\sim$ is the equivalence relation $G(\preceq) \cap G(\preceq^{-1})$ on X, and $\leq$ is the partial order on $X/\sim$ defined by $[a] \leq [b]$ if and only if $a \preceq b$.*

(a) *Any $\preceq$-increasing or $\preceq$-decreasing set is $\sim$-saturated.*

(b) *The quotient map $q : X \to X/\sim$ carries increasing (decreasing) sets to increasing (decreasing) sets. Specifically, $q(i_{\preceq}(A)) = i_{\leq}(q(A))$ for any subset $A \subseteq X$, and dually.*

(c) *If $\preceq^*$ is a preorder on X with $G(\preceq) \subseteq G(\preceq^*)$ and $\sim_*$ is defined from $\preceq^*$ as $\sim$ is defined from $\preceq$, then the $\sim_*$-equivalence classes are $\sim$-saturated.*

Preorders induced by functions. Any continuous increasing function $f : (X, \tau, \leq) \to (Y, \gamma, \sqsubseteq)$ between two partially ordered topological spaces induces a preorder $\preceq_f$ on X defined by $a \preceq_f b$ if and only if $f(a) \sqsubseteq f(b)$. Furthermore, $G(\leq) \subseteq G(\preceq_f)$.

Suppose $(Y, \gamma, \sqsubseteq)$ is T_i-ordered for some $i \in \{0, 1, 2, S2\}$. Now $G(\preceq_f) = (f^{-1} \times f^{-1})(\sqsubseteq)$. Noting that f^{-1} and $f^{-1} \times f^{-1}$ carry open (respectively, closed, $\sqsubseteq$-increasing, $\sqsubseteq$-decreasing, disjoint) sets to open (respectively, closed, $\preceq_f$-increasing, $\preceq_f$-decreasing, disjoint) sets and that the T_i-(pre)ordered properties ($i \in \{0, 1, 2, S2\}$) are defined in terms of open/closed/increasing/decreasing/disjoint sets, it follows that $(X, \tau, \preceq_f)$ is T_i-preordered.

Also, if $\sim_f$ is the equivalence relation $G(\preceq_f) \cap G(\preceq_f^{-1})$, observe that the $\sim_f$-equivalence class of $a \in X$ is $[a] = f^{-1}(f(a))$, a fiber of the map f. Thus, $\preceq_f$ is a partial order if and only if f is injective.

We now apply the special quotient construction described above to the preorder $\preceq_f$ induced by a function f. Suppose $f : (X, \tau, \leq) \to (Y, \gamma, \sqsubseteq)$ is a continuous increasing function between partially ordered spaces $(X, \tau, \leq)$ and $(Y, \gamma, \sqsubseteq)$, $\preceq_f$ is the preorder on X defined by $a \preceq_f b$ if and only if $f(a) \sqsubseteq f(b)$, $\sim_f$ is the equivalence relation on X defined by $a \sim_f b$ if and only if $a \preceq_f b$ and $b \preceq_f a$, and $\leq_f$ is the partial order on $X/\sim_f$ defined by $[a] \leq_f [b]$ if and only if $a \preceq_f b$. We now show that if $(Y, \gamma, \sqsubseteq)$ is T_i-ordered for some $i \in \{0, 1, 2, S2\}$, then the partially ordered space $(X/\sim_f, \tau/\sim_f, \leq_f)$ is also T_i-ordered. Define $h : (X/\sim_f, \tau/\sim_f, \leq_f) \to (Y, \gamma, \sqsubseteq)$ by $h([a]) = h(f^{-1}f(a)) = f(a)$, that is, $h([a]) = fq^{-1}[a]$ where $q : X \to X/\sim_f$ is the natural quotient map. Now h is continuous, for if U is open in Y, then $h^{-1}(U) = qf^{-1}(U)$ which is open since f is continuous and the quotient map carries saturated open sets to open sets. The definitions of $\leq_f, \preceq_f, h$, and $\preceq_h$ respectively imply the following implications:

$$[a] \leq_f [b] \iff a \preceq_f b \iff f(a) \sqsubseteq f(b) \iff h([a]) \sqsubseteq h([b]) \iff [a] \preceq_h [b].$$

Thus, h is increasing and $\preceq_h = \leq_f$. By the remarks of the second paragraph after Proposition 2.1, it follows that $(X/\sim_f, \tau/\sim_f, \leq_f)$ is T_i-ordered.

Proposition 2.2. *Suppose $(X, \tau, \preceq)$ is a preordered topological space, $\sim$ is the equivalence relation on X defined by $a \sim b$ if and only if $a \preceq b$ and $b \preceq a$, and $\leq$ is the partial order on $X/\sim$ defined by $[a] \leq [b]$ if and only if $a \preceq b$. Then for $i \in \{0, 1, S2\}$, $(X, \tau, \preceq)$ is T_i-preordered if and only if $(X/\sim, \tau/\sim, \leq)$ is T_i-ordered.*

Proof. Suppose $i \in \{0, 1, S2\}$ and $(X/\sim, \tau/\sim, \leq)$ is T_i-ordered. The two paragraphs following Proposition 2.1 remain valid for a function f from a preordered space to a partially ordered space, and if f is taken to be the quotient map q from $(X, \tau, \preceq)$ to $(X/\sim, \tau/\sim, \leq)$, then $\preceq_f = \preceq$. Thus, $(X, \tau, \preceq)$ is T_i-preordered.

For the converse, first suppose that $(X, \tau, \preceq)$ is T_0-preordered. If $[a] \neq [b]$ in $X/\sim$, then there is a $\preceq$-monotone open neighborhood N of one of the points a or b which does not contain the other. Now $q(N)$ is a $\leq$-monotone open neighborhood of one of the points $[a]$ or $[b]$ in $X/\sim$ which does not contain the other, so $X/\sim$ is T_0-ordered. Now suppose $(X, \tau, \preceq)$ is T_1-preordered. For $[x] \in X/\sim$, we have $i_{\leq}([x]) = i_{\leq}(q(x)) = q(i_{\preceq}(x))$ by Proposition 2.1 (b). Since $i_{\preceq}(x)$ is closed and saturated, $q(i_{\preceq}(x))$ will be closed in $X/\sim$. With the dual argument, this shows that $X/\sim$ is T_1-ordered. Finally, suppose $(X, \tau, \preceq)$ is T_{S2}-preordered. If $[a] \not\leq [b]$ in $X/\sim$, then $a \not\preceq b$ in X, so there exist a $\preceq$-increasing τ-open neighborhood N_a of a and a $\preceq$-decreasing τ-open neighborhood N_b of b in X which are disjoint. By Proposition 2.1 (a) and (b), it follows that $q(N_a)$ and $q(N_b)$ are the required $\leq$-monotone $\tau/\sim$-open neighborhoods separating $[a]$ and $[b]$ in $X/\sim$. □

The result of Proposition 2.2 does not hold for $i = 2$. While the reasoning of the first paragraph of the proof shows that if $(X/\sim, \tau/\sim, \leq)$ is T_2-ordered then $(X, \tau, \preceq)$ is T_2-preordered, the example below shows that the converse fails.

Example 2.3. If $(X, \tau, \preceq)$ is a T_2-preordered space, $\sim$ is the equivalence relation on X defined by $a \sim b$ if and only if $a \preceq b$ and $b \preceq a$, and $\leq$ is the partial order on $X/\sim$ defined by $[a] \leq [b]$ if and only if $a \preceq b$, then $(X/\sim, \tau/\sim, \leq)$ need not be T_2-ordered.

Let γ be the Euclidean topology on $\mathbb{R}$. Define a topology τ on $X = \mathbb{R}$ as follows: Each point of $\mathbb{Q}\setminus\{0\}$ is isolated. For $x \in (\mathbb{R}\setminus\mathbb{Q}) \cup \{0\}$, a τ-neighborhood of x is $\{x\} \cup (U \cap \mathbb{Q})$ where U is a γ-neighborhood of x.

Define a preorder $\preceq$ on X by $a \preceq b$ if and only if $a = b$ or $\{a, b\} \subseteq \mathbb{R} \setminus \mathbb{Q}$. (In fact, $\preceq$ is already an equivalence relation, so $\sim = \preceq$.) The graph of $\preceq$ is $(\mathbb{R} \setminus \mathbb{Q})^2 \cup \Delta_X$. Because each γ-neighborhood of $x \in X$ contains a τ-neighborhood of x, it follows that (X, τ) is T_2, and thus Δ_X is closed in $X \times X$. Observe that $\mathbb{Q}$ is a neighborhood of each of its points, so $\mathbb{R} \setminus \mathbb{Q}$ is closed in X. It follows that $G(\preceq) = (\mathbb{R} \setminus \mathbb{Q})^2 \cup \Delta_X$ is closed in $X \times X$, so $\preceq$ is T_2-preordered.

To see that $(X/\sim, \tau/\sim, \leq)$ is not T_2-ordered, suppose to the contrary that it is. Now $\pi \not\preceq 0$ in X, so $[\pi] \not\leq [0]$ in $X/\sim$, and thus there exist disjoint sets M, N, where M is a $\leq$-increasing $\tau/\sim$-neighborhood of $[\pi]$ and N is a $\leq$-decreasing $\tau/\sim$-neighborhood of $[0]$. If $q : X \to X/\sim$ is the quotient map, then $q^{-1}(N)$ contains a τ-neighborhood $\{0\}\cup(B\cap\mathbb{Q})$ of 0, where B is a γ-open neighborhood of 0. For any $b\in B\setminus\mathbb{Q}$, we have $b\in\mathbb{R}\setminus\mathbb{Q}=[\pi]\subseteq q^{-1}(M)$. Now $q^{-1}(M)$ is a $\preceq$-increasing τ-neighborhood of b. But a $\preceq$-increasing τ-neighborhood of $b\in\mathbb{R}\setminus\mathbb{Q}$ has form $(\mathbb{R}\setminus\mathbb{Q})\cup U$ where U is a γ-neighborhood of b. Now $U\cap B\neq\varnothing\Rightarrow U\cap(B\cap\mathbb{Q})\neq\varnothing\Rightarrow q^{-1}(M)\cap q^{-1}(N)\neq\varnothing$, contrary to $M\cap N=\varnothing$. Thus, $(X/\sim,\tau/\sim,\leq)$ is not T_2-ordered.

The existence construction. Suppose $(X,\tau,\leq)$ is a given partially ordered topological space. For $i\in\{0,1,2,S2\}$, let

$$\mathcal{P}_i=\{\preceq^*\ :\ \preceq^*\text{ is a preorder on }X,\ G(\leq)\subseteq G(\preceq^*),\ \text{and }(X/\sim_*,\tau/\sim_*,\leq^*)\text{ is }\mathrm{T_i}\text{-ordered}\}.$$

Note that $\mathcal{P}_i\neq\varnothing$ since $X\times X$ belongs to it. If $i\in\{0,1,S2\}$, by Proposition 2.2, we have

$$\mathcal{P}_i=\{\preceq^*: G(\leq)\subseteq G(\preceq^*)\text{ and }(X,\tau,\preceq^*)\text{ is }\mathrm{T_i}\text{-preordered}\}.$$

Let $G(\preceq^i)=\bigcap\mathcal{P}_i$. As an intersection of preorders containing $G(\leq)$, $\preceq^i$ is also a preorder containing $G(\leq)$.

Proposition 2.4. *For $i\in\{0,1,2,S2\}$, $(X/\sim_i,\tau/\sim_i,\leq^i)$ is T_i-ordered.*

Proof. *$i=0$:* Suppose $[x]\neq[y]$ in $(X/\sim_0,\tau/\sim_0,\leq^0)$. Then $x\not\sim_0 y$ in X, so for some $\preceq^*\in\mathcal{P}_0$, there exists a $\preceq^*$-monotone open neighborhood N of a which does not contain b, where $\{a,b\}=\{x,y\}$. Applying Proposition 2.1 (b), we obtain a $\leq^*$-monotone open neighborhood N' of $[a]_{\sim_*}$ which does not contain $[b]_{\sim_*}$. Since $G(\leq^0)\subseteq G(\preceq^*)$, Proposition 2.1 (c) implies the existence of a natural increasing quotient map $q: X/\sim_0\to X/\sim_*$, and $q^{-1}(N')$ is a $\leq^0$-monotone open neighborhood of $[a]_{\sim_0}$ which does not contain $[b]_{\sim_0}$. Thus, $(X/\sim_0,\tau/\sim_0,\leq^0)$ is T_0-ordered.

$i=1$: Because the increasing hull of x in $\preceq^1=\bigcap\mathcal{P}_1$ is the intersection of the increasing hulls of x in each $\preceq^*$ in $\mathcal{P}_1$, and each of these latter increasing hulls is closed, it follows that $i_{\preceq^1}(x)$ is closed for any $x\in X$. With the dual argument, we have $(X,\tau,\preceq^1)$ is T_1-preordered. By Proposition 2.2, $(X/\sim_1,\tau/\sim_1,\leq^1)$ is T_1-ordered.

$i=2$: Suppose $[a]\not\leq[b]$ in $(X/\sim_2,\tau/\sim_2,\leq^2)$. Then there exists $\preceq^*\in\mathcal{P}_2$ such that $[a]_{\sim_*}\not\leq^*[b]_{\sim_*}$ in the T_2-ordered space $(X/\sim_*,\tau/\sim_*,\leq^*)$. Let N_a and N_b be disjoint $\tau/\sim_*$ neighborhoods of $[a]_{\sim_*}$ and $[b]_{\sim_*}$ respectively, with N_a being $\leq^*$-increasing and N_b being $\leq^*$-decreasing. Since $G(\leq^2)\subseteq G(\preceq^*)$, the natural quotient map q from $X/\sim_2$ to $X/\sim_*$ yields $q^{-1}(N_a)$ and $q^{-1}(N_b)$ as oppositely directed monotone neighborhoods separating $[a]$ and $[b]$ in $X/\sim_2$. Thus, $(X/\sim_2,\tau/\sim_2,\leq^2)$ is T_2-ordered.

Taking N_a and N_b above to be open sets proves the case $i=S2$. □

We are now ready for the main result of this section.

Theorem 2.5. *Suppose $(X,\tau,\leq)$ is a partially ordered topological space and $i\in\{0,1,2,S2\}$. Then the T_i-ordered reflection of $(X,\tau,\leq)$ is $(X/\sim_i,\tau/\sim_i,\leq^i)$.*

Proof. Suppose $i \in \{0, 1, 2, S2\}$ is given, $(Y, \gamma, \sqsubseteq)$ is T_i-ordered, and $f : (X, \tau, \leq) \to (Y, \gamma, \sqsubseteq)$ is continuous and increasing. From the paragraph preceding Proposition 2.2, it follows that $(X/\sim_f, \tau/\sim_f, \leq_f)$ is a T_i-ordered space with $G(\leq) \subseteq G(\preceq_f)$. From the definition of $\preceq^i$, we have $G(\preceq^i) \subseteq G(\preceq_f)$. From Proposition 2.1 (c), there is a natural continuous increasing quotient map $q : (X/\sim_i, \tau/\sim_i, \leq^i) \to (X/\sim_f, \tau/\sim_f, \leq_f)$ which carries $[a]_{\sim_i}$ to $[a]_{\sim_f}$. We have shown above that there is a continuous increasing function $h : (X/\sim_f, \tau/\sim_f, \leq_f) \to (Y, \gamma, \sqsubseteq)$. Now $hq : X/\sim_i \to Y$ is continuous and increasing. Thus, each continuous increasing function $f : X \to Y$ can be lifted through $X/\sim_i$, and from the construction, this lifting is unique. Thus, $X/\sim_i$ is the T_i-ordered reflection of X. □

It is easy to verify that this construction gives the property Q reflection of $(X, \tau, \leq)$ as a quotient for any property Q for which

(a) $(X/\sim_Q, \tau/\sim_Q, \leq^Q)$ has property Q where $a \sim_Q b$ if and only if $a \preceq_Q b$ and $b \preceq_Q a$; $G(\preceq_Q) = \bigcap\{G(\preceq^*) : G(\leq) \subseteq G(\preceq^*)$ and $(X/\sim_*, \tau/\sim_*, \leq^*)$ is an ordered space with property Q$\}$; and $[a] \leq^Q [b]$ in $X/\sim_Q$ if and only if $a \preceq_Q b$ in X.

(b) If $(Y, \gamma, \sqsubseteq)$ has property Q and $f : (X, \tau, \leq) \to (Y, \gamma, \sqsubseteq)$ is continuous and increasing, then $(X/\sim_f, \tau/\sim_f, \leq_f)$ has property Q.

Furthermore, the methods of this section can be used to find the T_i reflection $(i = 0, 1, 2)$ of a topological space (X, τ) by considering (X, τ) as a discretely ordered topological space $(X, \tau, =)$ and taking all preorders to be equivalence relations, so that the resulting quotients are discretely ordered.

3. The T_0-ordered reflection

The construction of the T_i-ordered reflections in the previous section was an extrinsic construction—working from outside the space $(X, \tau, \leq)$—which produced the T_i-ordered reflection as a quotient based on the intersection of all suitable preorders on X for which the indicated quotient construction would yield a T_i-ordered space. In this section, we present an intrinsic construction of the T_0-ordered reflection and discuss some other properties of the T_0-ordered reflection. Intrinsic constructions of the other T_i-ordered reflections $(i > 0)$ studied in the previous section appear to be much more complicated.

In a T_0-ordered space, $D(x) = D(y)$ and $I(x) = I(y)$ would imply $x = y$. If our space is not T_0-ordered, then there may be distinct elements x and y with $D(x) = D(y)$ and $I(x) = I(y)$. Our strategy will be to say two such points are equivalent and mod out by this equivalence relation.

Suppose $(X, \tau, \leq)$ is an ordered topological space. For $x, y \in X$, define $x \approx y$ if and only if $D(x) = D(y)$ and $I(x) = I(y)$. Order the set $X/\approx$ of $\approx$-equivalence classes by the *finite step order:*

$$[z_0] \leq^0 [z_n] \iff \exists [z_1], [z_2], \ldots, [z_{n-1}] \text{ and } \exists z_i', z_i^* \in [z_i] (i = 0, 1, \ldots, n) \text{ with } z_i' \leq z_{i+1}^* \ \forall i = 0, 1, \ldots, n-1.$$

First note that this is indeed antisymmetric and therefore is a partial order: Suppose $[z_0] \leq^0 [z_n]$ and $[z_n] \leq^0 [z_0]$. Then there exist $[z_i]$ $(i = 1, \ldots, n, \ldots, m)$ with $[z_0] = [z_m]$ and there exist $z_i', z_i^* \in [z_i]$ such that $z_i' \leq z_{i+1}^*$ for all $i = 0, \ldots, m-1$. To show $[z_0] = [z_n]$, suppose not. Then either $D(z_0) \neq D(z_n)$ or $I(z_0) \neq I(z_n)$. Now

$$z_i' \leq z_{i+1}^* \Rightarrow z_{i+1}^* \in I(z_i') \Rightarrow I(z_{i+1}^*) \subseteq I(z_i') \Rightarrow I(z_{i+1}) \subseteq I(z_i).$$

Applying this for $i = 0, \ldots, m-1$ gives $I(z_0) \supseteq I(z_1) \supseteq \cdots \supseteq I(z_m) = I(z_0)$. Thus, $I(z_i) = I(z_0)$ $\forall i \in \{1, \ldots, m\}$. Dually, $D(z_i) = D(z_0)$ $\forall i \in \{1, \ldots, m\}$. It follows that $z_i \approx z_0$ $\forall i \in \{1, \ldots, m\}$, so $[z_0] = [z_n]$, and $\leq$ is antisymmetric.

In fact, the argument above shows that

$$[x] \leq^0 [y] \Rightarrow I(y) \subseteq I(x) \text{ and } D(x) \subseteq D(y).$$

At this point, one can verify that the equivalence relation $\approx$ agrees with $\sim_0$ introduced in the previous section and that the finite step order described above agrees with the order $\leq^0$ defined in the previous section by $[a] \leq^0 [b]$ if and only if $a \preceq^0 b$ where $G(\preceq^0) = \bigcap \mathcal{P}_0$, and thus the T_0-ordered reflection of $(X, \tau, \leq)$ is $(X/\approx, \tau/\approx, \leq^0)$. However, we will continue our intrinsic approach and prove this directly.

It is easy to show that any closed or open monotone set in X is $\approx$-saturated and that the quotient map $f : X \to X/\approx$ carries closed increasing sets to closed increasing sets and open increasing sets to open increasing sets. The dual statement (obtained by replacing "increasing" by "decreasing") also holds. It follows that f is an *ordered quotient map* as defined in Definition 6 of [6]. It is easily verified that if $\mathcal{D} = \{f^{-1}(y) : y \in X/\approx\}$ is the decomposition of X associated with the quotient map $f : X \to X/\approx$, then for each $[x] \in \mathcal{D}$ and each increasing (decreasing) open set U containing $[x]$, there exists a saturated increasing (decreasing) open set containing $[x]$ which is contained in U. We have A is closed and increasing in X if and only if $f(A)$ is closed and increasing in $X/\approx$, and B is closed and increasing in $X/\approx$ if and only if $f^{-1}(B)$ is closed and increasing in X. Furthermore, because $I(x) = \bigcap \mathcal{C}$ where $\mathcal{C}$ is the collection of closed increasing sets containing x and $f(\bigcap \mathcal{C}) = \bigcap f(\mathcal{C})$ for any collection $\mathcal{C}$ of $\approx$-saturated sets, it follows that $f(I(x)) = I_{X/\approx}([x])$. Dually, $f(D(x)) = D_{X/\approx}([x])$.

Theorem 3.1. *Suppose $(X, \tau, \leq)$ is a partially ordered topological space, and $a \approx b$ if and only if $D(a) = D(b)$ and $I(a) = I(b)$. Then $X/\approx$ with the quotient topology and the finite-step order is the T_0-ordered reflection of X.*

Proof. First we will show that $X/\approx$ is T_0-ordered. Suppose $I_{X/\approx}([x]) = I_{X/\approx}([y])$ and $D_{X/\approx}([x]) = D_{X/\approx}([y])$. If $f : X \to X/\approx$ is the natural ordered quotient map, then we have $f(I(x)) = f(I(y))$ and $f(D(x)) = f(D(y))$. Applying f^{-1} to the equalities above and recalling that $I(x)$ and $D(x)$ are saturated, we have $I(x) = I(y)$ and $D(x) = D(y)$, which implies $[x] = [y]$. Thus, $X/\approx$ is T_0-ordered.

Now suppose Y is any T_0-ordered space and $g : X \to Y$ is continuous and increasing. We will show that $\{g^{-1}(y) : y \in Y\}$ is saturated with respect to $\mathcal{D} = \{f^{-1}([x]) : [x] \in X/\approx\}$. Suppose to the contrary that there exists $y \in Y$ such that $g^{-1}(y)$ is not $\mathcal{D}$-saturated. Then there exist $b \in g^{-1}(y)$ and $a \in X \setminus g^{-1}(y)$ such that $[a] = [b]$ (that is, $f(a) = f(b)$).

Now $g^{-1}(I_Y(g(b))$ is a closed increasing set in X which contains $g^{-1}(g(b))$ and therefore contains b. But

$$\begin{aligned}
[a] = [b] \quad &\Rightarrow \quad I(a) = I(b) \\
&\Rightarrow \quad a \text{ is an element of every closed increasing set containing } b \\
&\Rightarrow \quad a \in g^{-1}(I_Y(g(b)) \\
&\Rightarrow \quad g(a) \in I_Y(g(b)) \\
&\Rightarrow \quad I_Y(g(a)) \subseteq I_Y(g(b)).
\end{aligned}$$

Repeating the argument of the last paragraph with a and b interchanged shows the reverse inclusion, so $I_Y(g(a)) = I_Y(g(b))$. The dual argument shows that $D_Y(g(a)) = D_Y(g(b))$. Since Y is T_0-ordered, this implies $g(a) = g(b)$, contrary to $a \in X \setminus g^{-1}(y)$ and $b \in g^{-1}(y)$.

Now since $\{g^{-1}(y) : y \in Y\}$ is $\mathcal{D}$-saturated, there is a natural quotient map h from $X/\mathcal{D} = X/\approx$ to Y, and $hf = g$. From the definition of the finite step order on $X/\approx$, it is clear that h is increasing, and h is clearly unique from the construction. Thus, $X/\approx$ is the T_0-ordered reflection of X. □

The theorem below characterizes those spaces whose T_0-ordered reflections are T_1-ordered. Similar results for the non-ordered setting can be found in [1], where a $T_{(i,j)}$-space is defined to be one whose T_i-reflection satisfies the T_j separation axiom ($0 \leq i < j \leq 2$). Comparing Theorem 3.5(iv) of [1] with Theorem 2(b) of [2], we note that $T_{(0,1)}$-spaces have been studied by Davis and others subsequently under the name of R_0-spaces.

Theorem 3.2. *The following are equivalent.*
(a) The T_0-ordered reflection $X/\approx$ of X is T_1-ordered.
(b) $[x] \not\leq^0 [y]$ in $X/\approx$ implies there exists an open increasing neighborhood of x not containing y and there exists an open decreasing neighborhood of y not containing x.
(c) $i([x]) = \bigcap\{N : N$ is an open increasing neighborhood of $x\}$ for any $x \in X$, and
$d([x]) = \bigcap\{N : N$ is an open decreasing neighborhood of $x\}$ for any $x \in X$.

Proof. (a) ⇒ (c): Because closed or open increasing sets are saturated, we have $i([x]) \subseteq \bigcap\{N : N$ is an open increasing neighborhood of $x\}$. Suppose $M = \bigcap\{N : N$ is an open increasing neighborhood of $x\} \not\subseteq i([x])$. Then there exists $y \in M \setminus i([x])$, and since M is saturated, $[y] \not\subseteq i([x])$. In particular, $[x] \not\leq^0 [y]$ in the T_1-ordered space $X/\approx$, so there exists an increasing open neighborhood J of $[x]$ in $X/\approx$ disjoint from $[y]$. Now if $f : X \to X/\approx$ is the quotient map, $f^{-1}(J)$ is an open increasing neighborhood of x disjoint from y. This contradicts $y \in M$. This proves that $i([x]) = \bigcap\{N : N$ is an open increasing neighborhood of $x\}$ for any $x \in X$. The other statement is proved dually.

$(c) \Rightarrow (a)$: Suppose (c). If $X/\approx$ is not T_1-ordered, then there exist $[x] \not\lesssim^0 [y]$ such that either (i) every increasing open neighborhood of $[x]$ in $X/\approx$ contains $[y]$, or (ii) every decreasing open neighborhood of $[y]$ in $X/\approx$ contains $[x]$. If (i) holds, then $[y] \in \bigcap\{N : N \text{ is an open increasing neighborhood of } x\} = i([x])$, contrary to $[x] \not\lesssim^0 [y]$. If (ii) holds, then $[x] \in \bigcap\{N : N \text{ is an open decreasing neighborhood of } y\} = d([y])$, contrary to $[x] \not\lesssim^0 [y]$.

$(a) \Rightarrow (b)$: Suppose (a). Now $[x] \not\lesssim^0 [y]$ in $X/\approx$ implies there exists an open increasing (respectively, decreasing) neighborhood of $[x]$ (respectively, $[y]$) not containing $[y]$ (respectively, $[x]$). Taking f^{-1} of these neighborhoods gives the desired neighborhoods in X.

$(b) \Rightarrow (a)$: If $[x] \not\lesssim^0 [y]$ in $X/\approx$, then by (b) there exists an open increasing neighborhood N of x not containing y and there exists an open decreasing neighborhood M of y not containing x. Now M and N are saturated, and since f is an ordered quotient map, $f(M)$ and $f(N)$ are monotone open neighborhoods of $[y]$ and $[x]$, respectively, which show that $X/\approx$ is T_1-ordered. □

A set A which satisfies $A = I(A) \cap D(A)$ is called a *c-set.* In [4], maximal filters of c-sets are used to construct the Wallman ordered compactification of an ordered space with convex topology. The Wallman ordered compactification w_0X is a universal compact T_1 extension. In [5], conditions involving c-sets are given to insure w_0X is T_1-ordered. Thus, one might expect c-sets to play a role in the T_1-ordered or even T_0-ordered reflection. Let $C(A) = I(A) \cap D(A)$, that is, let $C(A)$ be the smallest c-set containing A.

Proposition 3.3. *Suppose $(X, \tau, \leq)$ is an ordered topological space and let $\approx$ be the equivalence relation on X defined by $x \approx y$ if and only if $D(x) = D(y)$ and $I(x) = I(y)$. Then $x \approx y$ if and only if $C(x) = C(y)$.*

Proof. Suppose $C(x) = C(y)$. Then $x \in C(y) \subseteq I(y)$, so $I(x) \subseteq I(y)$. Interchanging x and y shows that $I(y) \subseteq I(x)$, so $I(x) = I(y)$. Dually, $D(x) = D(y)$, so $x \approx y$. The converse is immediate. □

Thus, the equivalence classes of the T_0-ordered reflection are determined by the closure operator $C(\cdot)$. If X has a convex topology, this closure operator is especially nice.

Theorem 3.4. *If the ordered topological space $(X, \tau, \leq)$ has a convex topology, then the topological space (X', τ') underlying its T_0-ordered reflection $(X', \tau', \leq')$ is simply the T_0 reflection of (X, τ).*

Proof. Suppose X has a convex topology. We will show that $C(x) = I(x) \cap D(x) = cl\{x\}$. Clearly $y \in cl\{x\} \Rightarrow y \in I(x) \cap D(x)$. For the converse, suppose $y \notin cl\{x\}$. Then there exist an increasing open neighborhood N_y of y and a decreasing open neighborhood M_y of y with $x \notin N_y \cap M_y$. Thus, either $x \notin N_y$ or $x \notin M_y$, and taking complements shows that $y \notin D(x)$ or $y \notin I(x)$, that is, $y \notin I(x) \cap D(x)$, as needed.

By Proposition 3.3, $x \approx y$ if and only if $cl\{x\} = cl\{y\}$. It is well-known that the T_0 reflection of (X, τ) is given by the quotient topology on the quotient set $X/\simeq$ where $x \simeq y$ if and only if $cl\{x\} = cl\{y\}$. □

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