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SEPARATION PROPERTIES OF THE WALLMAN ORDERED COMPACTIFICATION

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ABSTRACT. The Wallman ordered compactification w_0X of a topological ordered space X is T_2-ordered (and hence equivalent to the Stone-Čech ordered compactification) iff X is a T_4-ordered c-space. In particular, these two ordered compactifications are equivalent when X is n dimensional Euclidean space iff $n \leq 2$. When X is a c-space, w_0X is T_1-ordered; we give conditions on X under which the converse statement is also true. We also find conditions on X which are necessary and sufficient for w_0X to be T_2. Several examples provide further insight into the separation properties of w_0X.

KEY WORDS AND PHRASES. c-set, maximal c-filter, T_1-ordered space, T_2-ordered space, ordered compactification.

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Introduction.

The Wallman ordered compactification w_0X of a T_1-ordered space X was introduced in 1979 by Choe and Park [1]. In [3] one of the authors showed (in the terminology of this paper) that w_0X is T_2-ordered iff X is a T_4-ordered c-space, and that for such spaces, w_0X is equivalent to the Stone-Čech ordered (or Nachbin) compactification β_0X of X.

This paper continues the study of the separation properties of w_0X. If X is a c-space (meaning that the increasing and decreasing hulls of every c-set are closed), then w_0X is T_1-ordered, and under certain further restrictions on X the condition of being a c-space is shown to be necessary in order for w_0X to be T_1-ordered (see Theorems 2.7 and 2.8). Two conditions on X are found which are necessary and sufficient for w_0X to be T_2; one is an ultrafilter condition, while the other is a version of normality for ordered spaces which we call "c-normally ordered." For T_1-ordered c-spaces, the notions "c-normally ordered" and "normally ordered" (as defined by Nachbin, [5]) are equivalent, but for T_1-ordered spaces in general it is shown by examples that neither property implies the other.

One motivation for studying the Wallman ordered compactification is that it gives a convenient filter characterization for β_0X when X is a T_4-ordered c-space. For Euclidean n-space R^n, we show that w_0R^n and β_0R^n are equivalent iff $n \leq 2$, and we then give a description of β_0R^2 based on the Wallman characterization of compactification points in β_0R^2 as non-convergent maximal c-filters. Other examples are given to show how the separation properties of the Wallman ordered compactification can fail in various ways and combinations.

1. The Wallman Ordered Compactification.

If $(X, \leq)$ is a poset and A a non-empty subset of X, we define $d(A) = \{y \in X : y \leq x \text{ for some } x \in A\}$ to be the *decreasing hull* of A; the *increasing hull* $i(A)$ is defined dually. We shall write $d(x)$ ($i(x)$) in place of $d(\{x\})$ ($i(\{x\})$). A subset A is *increasing* (respectively, decreasing) if $A = i(A)$ (respectively, $A = d(A)$). A set which is either increasing or decreasing is said to be *monotone*. If $A = i(A) \cap d(A)$, then A is called a *convex* set.

We shall use the term *space* throughout this paper to mean a triple $(X, \leq, \tau)$, where $(X, \leq)$ is a poset and τ a *convex* toplogy on X (i.e., a topology for which the open monotone sets constitute an open subbase). When there is no danger of confusion, we shall designate the space $(X, \leq, \tau)$ simply by "X".

For any space X, we shall use the term *fundamental open set* to mean any set expressible as a finite intersection of finite unions of monotone open sets. The set $\mathcal{U}_X$ of all fundamental open sets forms an open base for X. The complement of a fundamental open set will be called a *fundamental closed set.*

Let A be a subset of a space X, and let $I(A)$ (respectively, $(D(A))$ be the smallest closed and increasing (respectively, closed and decreasing) set containing A, and let $A^\wedge = I(A) \cap D(A)$. If $A = A^\wedge$ then A is called a *c-set*; let C_X denote the collection of all c-sets on a space X. One can verify that C_X is closed under intersections and forms a subbase for the collection of closed sets in a space X. The relationship between fundamental open sets and c-sets can be described as follows.

Proposition 1.1 Let X be a space. Then $U \in \mathcal{U}_X$ iff $X - U$ is a finite union of c-sets.

If $\mathcal{F}$ is a filter on a space X, let $I(\mathcal{F})$ be the filter generated by $\{I(F) : F \in \mathcal{F}\}$; the filters $D(\mathcal{F}), i(\mathcal{F})$, and $d(\mathcal{F})$ are defined similarly. The fixed ultrafilter generated by an element x in X is denoted by $\mathring{x}$. If $\mathcal{F}$ and $\mathcal{G}$ are filters on X which do not contain disjoint sets, let $\mathcal{F} \vee \mathcal{G}$ designate the filter generated by $\{F \cap G : F \in \mathcal{F}, G \in \mathcal{G}\}$; if $F \cap G = \phi$ for some $F \in \mathcal{F}$ and $G \in \mathcal{G}$, we say that $\mathcal{F} \vee \mathcal{G}$ "fails to exist" (as a proper filter).

For any filter $\mathcal{F}$, the filter $\mathcal{F}^{\wedge} = I(\mathcal{F}) \vee D(\mathcal{F})$ exists and is generated by $\{F^{\wedge} : F \in \mathcal{F}\}$. If $\mathcal{F} = \mathcal{F}^{\wedge}$, then $\mathcal{F}$ is called a *c-filter*. It is easy to show (using Zorn's Lemma) that every c-filter is coarser than a maximal c-filter. In our study of the Wallman ordered compactification, which is based on maximal c-filters, the following characterization will be useful.

Proposition 1.2 A c-filter $\mathcal{F}$ on a space X is maximal iff, for each c-set A, either $A \in \mathcal{F}$ or $X - A \in \mathcal{F}$.

Proof. If $\mathcal{F}$ is a maximal c-filter and $A \notin \mathcal{F}$, then $\mathcal{F}$ can have no trace on A, for such a trace would be a c-filter strictly finer than A. Thus $X - A \in \mathcal{F}$. Conversely, if $\mathcal{F}$ is a c-filter which is not maximal, and $\mathcal{G}$ is a strictly finer c-filter, then some c-set G in $\mathcal{G}$ has the property that neither G nor $X - G$ is in $\mathcal{F}$, contrary to the stated condition. ∎

A space X is T_1*-ordered* [4] if, for each $x \in X, i(x)$ and $d(x)$ are both closed sets. Note that in a T_1-ordered space, each singleton $\{x\}$ is a c-set. A space with closed order is defined to be T_2-ordered [4]. A space X is *normally ordered* [5] if, whenever A and B are disjoint closed sets, with A increasing and B decreasing, there are disjoint open sets U and V, with U increasing and V decreasing, such that $A \subsetneq U$ and $B \subseteq V$. A space which is both normally ordered and T_1-ordered is said to be T_4*-ordered* [3]. Priestly, [6], defined a *C-space* to be one in which $i(A)$ and $d(A)$ are closed whenever A is closed. We define a *c-space* to be one in which every c-set A has the property that $i(A)$ and $d(A)$ are closed sets. Obviously, every C-space is a c-space; in particular, the compact, T_2-ordered spaces are c-spaces. An alternate characterization for c-spaces is given in the next proposition (see [3]).

Proposition 1.3 If X is a c-space and A, B are c-sets, then $I(A) \cap B = \phi$ implies $I(A) \cap D(B) = \phi$, and $D(A) \cap B = \phi$ implies $D(A) \cap I(B) = \phi$. If X is T_4-ordered and the two preceding implications hold for arbitrary c-sets A and B, then X is a c-space.

The Wallman ordered compactification can be constructed for any T_1-ordered space X. The original construction by Choe and Park [1] was based on "maximal bifilters"; we shall follow the approach of [3] in which maximal c-filters form the underlying set for w_0X. Given a T_1-ordered space X, let $w_0X = \{\mathring{x}: x \in X\} \cup X'$, where X' is the set of all non-convergent maximal c-filters. A partial order relation is defined for w_0X as follows: $\mathcal{F} \lesssim \mathcal{G}$ iff $I(\mathcal{F}) \subseteq \mathcal{G}$ and $D(\mathcal{G}) \subseteq \mathcal{F}$. The embedding map $\varphi : X \to w_0X$ given by $\varphi(x) = \mathring{x}$ for all x in X is obviously increasing.

For any subset A of X, let $A^* = \{\mathcal{F} \in w_0X : A \in \mathcal{F}\}$. If $\mathcal{F}$ is a filter on X, let $\mathcal{F}^*$ be the filter on w_0X generated by $\{F^* : F \in \mathcal{F}\}$. The fact that the latter collection is a filter base and other important properties of this set operator follow from the next proposition.

Proposition 1.4 Let X be a T_1-ordered space.

(a) For all subsets A, B of X, $(A \cap B)^* = A^* \cap B^*$

(b) If $A, B \in C_X$, then $((X - A) \cup (X - B))^* = (X - A)^* \cup (X - B)^*$

(c) If $A \in C_X$, then $(X - A)^* = X^* - A^*$.

Proof. Statement (a) is clear, and (b) follows from Proposition 1.2; (c) is an easy consequence of (b). ∎

The topology for w_0X is defined by choosing for a subbase of closed sets the collection $\{A^* : A \in C_X\}$. If $U \in \mathcal{U}_X$, then U is a finite intersection of complements of c-sets and U^* is open in w_0X; indeed $\{U^* : U \in \mathcal{U}_X\}$ is a base for the open sets in w_0X. In particular, sets of the form V^* where V is open and monotone in X form an open subbase for w_0X. It should be noted that if V is a non-fundamental open set in X, it is not generally true that V^* is open in w_0X. The following facts about the topology of w_0X will be stated for future reference.

Proposition 1.5 Let X be a T_1-ordered space.

(a) If B is a monotone closed (respectively, open) set in X, then B^* is monotone in the same sense and closed (respectively, open) in w_0X.

(b) If $\mathcal{F} \in w_0X$, then the neighborhood filter $\mathcal{V}^*(\mathcal{F})$ at $\mathcal{F}$ in w_0X has for its filter base $\{U^* : U \in \mathcal{F} \cap \mathcal{U}_X\}$.

The next two theorems summarize the main results already known about the Wallman ordered compactification. Proofs for these propositions form the main results of [1] and [3] and the reader is referred to these sources for further details. Here we should mention again that the proofs in [1] are formulated in the language of "bifilters", but the translation into "c-filter" terminology presents no difficulties.

Theorem 1.6 For any T_1-ordered space X, (w_0X, φ) is an ordered compactification of X, and w_0X is a T_1 topological space. Also, w_0X is T_2-ordered iff X is a T_4-ordered c-space.

Theorem 1.7 Let X be a T_1-ordered space, Y a T_2-ordered compact space, and $f : X \to Y$ a continuous increasing function. Then there is a unique continuous increasing function $\bar{f} : w_0X \to Y$ such that $\bar{f} \circ \varphi = f$.

Let us recall that for a space which admits a T_2-ordered compactification (see [5] for a characterization of such spaces) there is always a largest T_2-ordered compactification called the Stone-Čech ordered (or Nachbin) compactification denoted by β_0X (see [2], [5]). The two preceding theorems yield the following important corollary.

Corollary 1.8 For a space X which admits a T_2-ordered compactification, w_0X and β_0X are equivalent iff X is a T_4-ordered c-space.

2. Separation Properties of w_0X.

Given a T_1-ordered space X, we already know that w_0X is T_1, and that w_0X is T_2-ordered iff X is a T_4-ordered c-space. We shall now examine conditions on X subject to which w_0X is T_1-ordered or T_2. As it turns out, w_0X can fail to have either of these latter properties, can have either one without the other, or can have both properties and still fail to be T_2-ordered; examples are given later to illustrate all of these possibilities. We begin by finding conditions on X which are necessary and sufficient for w_0X to be T_2.

Proposition 2.1 Let $\mathcal{F}$ be an ultrafilter and $\mathcal{G}$ a maximal c-filter on X. Then $\varphi(\mathcal{F}) \to \mathcal{G}$ *in*w_0X iff $\mathcal{F}^\wedge \subseteq \mathcal{G}$.

Proof. Let $\varphi(\mathcal{F}) \to \mathcal{G}$ in w_0X. Let $F \in \mathcal{F}$ be a c-set. If $F \notin \mathcal{G}$ then either $I(F) \notin \mathcal{G}$ or $D(F) \notin \mathcal{G}$; without loss of generality, assume the former. Then $I(F) \notin \mathcal{G}$ implies, by Proposition 1.2, that $X - I(F) \in \mathcal{G}$, and therefore $\mathcal{G} \in (X - I(F))^*$, which is a subbasic open neighborhood of $\mathcal{G}$ in w_0X. Now $\varphi(\mathcal{F}) \to \mathcal{G}$ implies $(X - I(F))^* \in \varphi(\mathcal{F})$, and consequently $X - I(F) \in \mathcal{F}$. This contradicts the fact that $F \in \mathcal{F}$, and therefore every element of $\mathcal{F}^\wedge$ is in $\mathcal{G}$.

Conversely, let $\mathcal{F}^\wedge \subseteq \mathcal{G}$, and $(X-A)^*$ be a subbasic open neighborhood of $\mathcal{G}$, where A is closed and monotone in X. Since $\mathcal{F}$ is an ultrafilter, either $A \in \mathcal{F}$ or $X - A \in \mathcal{F}$. If $A \in \mathcal{F}$ then $A \in \mathcal{F}^\wedge$, which in turn implies $A \in \mathcal{G}$, contrary to the fact that $X - A \in \mathcal{G}$. Thus $X - A \in \mathcal{F}$, and hence $(X-A)^* \in \varphi(\mathcal{F})$. Since $(X-A)^*$ is an arbitrary subbasic open neighborhood of $\mathcal{G}$, $\varphi(\mathcal{F}) \to \mathcal{G}$. ∎

Theorem 2.2 Let X be a T_1-ordered space. Then w_0X is T_2 iff, for each ultrafilter $\mathcal{F}$ on X, there is a unique maximal c-filter $\mathcal{G}$ on X such that $\mathcal{F}^\wedge \subseteq \mathcal{G}$.

Proof. If w_0X is T_2 and $\mathcal{F}$ an ultrafilter on X, then $\varphi(\mathcal{F})$ is an ultrafilter on w_0X which must converge to some maximal c-filter $\mathcal{G}$, since w_0X is compact. By Proposition 2.1, $\mathcal{F}^\wedge \subseteq \mathcal{G}$. If there is a different maximal c-filter $\mathcal{H}$ with $\mathcal{F}^\wedge \subseteq \mathcal{H}$, then $\varphi(\mathcal{F})$ would also converge to $\mathcal{H}$, contrary to the assumption that w_0X is T_2. Thus $\mathcal{G}$ is unique.

Conversely, assume the uniqueness condition. If w_0X is not T_2, there is a filter $\mathcal{A}$ on w_0X converging to distinct elements $\mathcal{G}, \mathcal{H}$ in w_0X. Let $\mathcal{F}$ be an ultrafilter on X finer than the filter generated by $\{A \subseteq X : A^* \in \mathcal{A}\}$. One easily verifies that $\varphi(\mathcal{F})$ must converge to both $\mathcal{G}$ and $\mathcal{H}$, which, by Proposition 2.1, violates our assumed uniqueness condition. ∎

A space X is defined to be *c-normally ordered* if, for each pair of disjoint c-sets A, B, there are disjoint fundamental open sets U, V such that $A \subseteq U$ and $B \subseteq V$. As we shall see in later examples, there are spaces which are c-normally ordered but not normally ordered, and vice versa. Of course, both of these versions of "ordered normality" reduce to ordinary normality when the partial order for X is equality.

Theorem 2.3 The following conditions on a T_1-ordered space X are equivalent.

(1) X is c-normally ordered.

(2) Two disjoint fundamental closed sets in X can be separated by disjoint fundamental open neighborhoods.

(3) If A is a c-set in X, then every fundamental open set containing A contains a fundamental closed set which in turn contains a fundamental open neighborhood of A.

(4) For each ultrafilter $\mathcal{F}$ on X, there is a unique maximal c-filter $\mathcal{G}$ finer than $\mathcal{F}^\wedge$.

(5) w_0X is T_2.

Proof. The equivalence of (1), (2), and (3) is a routine exercise, and the equivalence of (4) and (5) was established in the previous theorem.

(1) ⇒ (5). If $\mathcal{F}$ and $\mathcal{G}$ are distinct maximal c-filters on X, then there are disjoint c-sets F in $\mathcal{F}$ and $G \in \mathcal{G}$. Let U and V be disjoint fundamental open neighborhoods of F and G respectively; then by Proposition 1.4, U^* and V^* are disjoint open neighborhoods of $\mathcal{F}$ and $\mathcal{G}$, respectively, in w_0X.

(5) ⇒ (1). Let A and B be disjoint c-sets in X. Then A^* and B^* are disjoint closed sets in w_0X, and since w_0X is compact and T_2, there are disjoint open sets M and N in w_0X such that $A^* \subseteq M$ and $B^* \subseteq N$. Since $\{U^* : U \in \mathcal{U}_X\}$ forms an open base for w_0X, there are subcollections $\{U_i^* : i \in I\}$ and $\{V_j^* : j \in J\}$ such that $M = \cup\{U_i^* : i \in I\}$ and $N = \cup\{V_j^* : j \in J\}$. Using the fact that A^* and B^* are compact subsets in w_0X, we can find finite subcovers $U_{i_1}^*, \cdots, U_{i_n}^*$ of A^* and $V_{j_1}^*, \cdots, V_{j_m}^*$ of B^*. Letting $U = U_{i_1} \cup \cdots \cup U_{i_n}$ and $V = U_{j_1} \cup \cdots \cup U_{j_m}$, we obtain disjoint fundamental open neighborhoods of A and B in X. ∎

Although neither of the properties "normally ordered" and "c-normally ordered" implies the other in general, the next theorem establishes the equivalence of these properties in T_1-ordered c-spaces. We first need the following lemma.

Lemma 2.4 Let X be a c-normally ordered c-space. If A is a c-set in X and U is an open, increasing neighborhood of A, then there is a closed, increasing neighborhood G of A such that $A \subseteq G \subseteq U$.

Proof. Let $B = X - U$; by Proposition 1.3, $I(A) \cap B = \phi$, and so $I(A)$ and B can be separated by disjoint, fundamental open sets W and V, respectively. By Proposition 1.1, $X - V$ is a finite union of c-sets $C_1, \cdots, C_n$. By Proposition 1.3, $I(C_i) \cap B = \phi$ for all indices i; let $G = \cup\{I(C_i) : i = 1, \cdots, n\}$. Thus G is closed and increasing, and $A \subseteq W \subseteq G \subseteq U$. ∎

Theorem 2.5 For a T_1-ordered c-space X, the following statements are equivalent.

(a) X is normally ordered.

(b) X is c-normally ordered.

(c) w_0X is T_2-ordered.

Proof. (a) ⇔ (c) is established in Theorem 1.6. (c) ⇒ (b) follows by Theorem 2.3. (b) ⇒ (c): It is sufficient to show that if $\mathcal{F}, \mathcal{G} \in w_0X$ and $\mathcal{F} \not\lesssim \mathcal{G}$, then there are disjoint neighborhoods of $\mathcal{F}$ and $\mathcal{G}$ in w_0X, where the former is an increasing set and the latter decreasing. If $\mathcal{F} \not\lesssim \mathcal{G}$, then either $I(\mathcal{F}) \not\subseteq \mathcal{G}$ or $D(\mathcal{G}) \not\subseteq \mathcal{F}$; without loss of generality, assume the latter. Since $\mathcal{F}$ is a maximal c-filter, there are c-sets $G \in \mathcal{G}$ and $F \in \mathcal{F}$ such that $D(G) \cap F = \phi$. By Lemma 2.4, there is a closed, increasing neighborhood N of F such that $N \cap D(G) = \phi$ and a fundamental open set W such that $F \subseteq W \subseteq N$. Now $(X - N)^*$, which is a decreasing open set in w_0X by Proposition 1.5, and the increasing hull of W^* in w_0X provide the desired neighborhoods which separate $\mathcal{F}$ and $\mathcal{G}$ in w_0X. ∎

If w_0X is T_2-ordered, then X is necessarily a T_1-ordered c-space; thus the following corollary is immediate.

Corollary 2.6 For a T_1-ordered space X, w_0X is T_2-ordered iff X is a c-normally ordered c-space.

Theorem 2.7 If X is a T_1-ordered c-space, then w_0X is T_1-ordered.

Proof. Let $\mathcal{F} \in w_0X$, and let $i_w(\mathcal{F}) = \{\mathcal{G} \in w_0X : \mathcal{F} \lesssim \mathcal{G}\}$ be the increasing hull of $\mathcal{F}$ in w_0X. Let "cl_w" denote the closure operator in w_0X. We shall show that $cl_w(i_w(\mathcal{F})) \subseteq i_w(\mathcal{F})$; a similar argument shows that the decreasing hull of $\mathcal{F}$ is closed, and hence that w_0X is T_1-ordered.

If $\mathcal{G} \in cl_w(i_w(\mathcal{F}))$, then for each $A \in C_X$ such that $\mathcal{G} \in (X-A)^*$, there is $\mathcal{H} \in i_w(\mathcal{F})$ such that $\mathcal{H} \in (X-A)^*$. With the help of Proposition 1.2, the last sentence may be restated as follows: if $\mathcal{G} \in cl_w(i_w(\mathcal{F}))$ and $A \in C_X$, then $A \notin \mathcal{G}$ implies there is $\mathcal{H} \in w_0X$ such that $\mathcal{F} \lesssim \mathcal{H}$ and $A \notin \mathcal{H}$.

Now assume that $\mathcal{G} \in cl_w(i_w(\mathcal{F}))$; if $\mathcal{G} \notin i_w(\mathcal{F})$, then either $I(\mathcal{F}) \not\subseteq \mathcal{G}$ or $D(\mathcal{G}) \not\subseteq \mathcal{F}$. Suppose $I(\mathcal{F}) \not\subseteq \mathcal{G}$; then there is $F \in \mathcal{F}$ such that $I(F) \notin \mathcal{G}$. But $\mathcal{G} \in cl_w(i_w(\mathcal{F}))$ implies there is $\mathcal{H} \in i_w(\mathcal{F})$ such that $I(F) \notin \mathcal{H}$, a contradiction. On the other hand, suppose $D(\mathcal{G}) \not\subseteq \mathcal{F}$. Since $\mathcal{F}$ is a maximal c-filter, there are c-sets $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $D(G) \cap F = \phi$, and by Proposition 1.3, $D(G) \cap I(F) = \phi$. Thus $I(F) \notin \mathcal{G}$, and a repetition of the preceding argument again yields a contradiction. We may therefore conclude that $\mathcal{G} \in i_w(\mathcal{F})$, and hence that $i_w(\mathcal{F})$ is closed. ∎

The converse of Theorem 2.7 does not hold in general, however the next theorem establishes a partial converse. We shall say that a net $(x_\lambda)_{\lambda \in \Lambda}$ in a space X is *upward directed* if, for each pair of indices $\lambda, \mu \in \Lambda$, there is $\sigma \in \Lambda$ such that $\lambda \leq \sigma$, $\mu \leq \sigma$, $x_\lambda \leq x_\sigma$, and $x_\mu \leq x_\sigma$. *Downward directed* nets are defined dually.

Theorem 2.8 Let X be a T_2-ordered space with the property that, whenever A is decreasing (respectively increasing) and $x \in cl_X A$, there is an upward directed (respectively, downward directed) net on A which converges to x. Then w_0X is T_1-ordered iff X is a c-space.

Proof. Suppose X is not a c-space. Then for some c-set A in X, either $i(A)$ is not closed or else $d(A)$ is not closed. There is no loss of generality in assuming the latter. Thus there is some $y \in cl_X d(A)$ such that $y \notin d(A)$; by assumption there is an upward directed net $(x_\lambda)_{\lambda \in \Lambda}$ on $d(A)$ converging to y. Then $\{i(x_\lambda) \cap A : \lambda \in \Lambda\}$ is a base for a c-filter $\mathcal{G}$ on X; let $\mathcal{F}$ be a maximal c-filter finer than $\mathcal{G}$.

Let $\mathcal{K}$ be the filter generated by the net $(x_\lambda)_{\lambda\in\Lambda}$. Then $\mathcal{K} \to y$, and if $\mathcal{F} \to x$ for some $x \in X$, then it must follow that $y \leq x$, since $\mathcal{K} \times \mathcal{F}$ has a trace on the order, and the order is closed. But $y \leq x$ is a contradiction since $A \in \mathcal{F}$ and A a c-set implies $x \in A$, and therefore $y \in d(A)$. Thus $\mathcal{F}$ must be a non-convergent maximal c-filter, and therefore an element of w_0X.

One may easily verify that $\mathring{x}_\lambda \lesssim \mathcal{F}$ holds for all $\lambda \in \Lambda$, but that $\mathring{y} \not\lesssim \mathcal{F}$. But $(x_\lambda)_{\lambda\in\Lambda} \to y$ in X implies $(\mathring{x}_\lambda)_{\lambda\in\Lambda} \to \mathring{y}$ in w_0X, and hence $\mathring{y} \in cl_w(d_w(\mathcal{F}))$, but $\mathring{y} \notin d_w(\mathcal{F})$. Thus w_0X is not T_1-ordered. ∎

Example 2.9 Let $X = A \cup B \cup \{a\} \cup \{b\}$, where $A = \{x_i : i = 1, 2, 3, \cdots\}$ and $B = \{y_i : i = 1, 2, 3, \cdots\}$. Define the topology for X by specifying that $\{x\}$ is open for $x \in A \cup B$; the neighborhood filter at a (respectively, b) is generated by sets of the form $A_n = \{x_i : i \geq n\}$ (respectively, $B_n = \{y_i : i \geq n\}$), where $n = 1, 2, 3, \cdots$. The order for X is the smallest partial order such that $x_i \leq y_i$ for each positive integer i.

It is evident from this construction that X is a compact, T_2, T_1-ordered space; thus we may identify X with w_0X. Note that every closed set in X is a c-set, and every open set is a fundamental open set; it follows that X is c-normally ordered. However X is not a c-space, since for the c-set $C = B \cup \{b\}, i(C) = A \cup C$ is not closed. Thus X is neither T_2-ordered nor normally ordered.

The main points illustrated by this example are that the converse of Theorem 2.7 does not hold in general, and that the conditions T_1-ordered, T_2, and c-normally ordered on X are not sufficient to guarantee that w_0X is T_2-ordered. This example also shows that a c-normally ordered space need not be normally ordered, even when the axioms T_1-ordered and T_2 are present.

3. Examples in Euclidean Space.

Additional insight into the behavior of the Wallman ordered compactification can be gained by study ing some simple examples in R^n (by which we mean Euclidean n-space with the usual product topology and product order), especially in the case $n = 2$. We shall show that $n = 2$ is the largest value of n for which w_0R^n is T_2-ordered, and hence the largest value of n for which $w_0R^n = \beta_0R^n$. We shall use the known properties w_0R^2 to describe β_0R^2. We also examine two simple subspaces of R^2 for which the Wallman ordered compactification is not T_2-ordered.

Theorem 3.1 Let A be a closed, convex subset of R^2. Then $i(A) = I(A)$ and $d(A) = D(A)$, and hence R^2 is a c-space.

Proof. We shall prove that $i(A)$ is closed; a similar argument shows that $d(A)$ is closed.

If $i(A)$ is not closed, then there is a sequence (x_n, y_n) in $i(A)$ such that $(x_n, y_n) \to (x_0, y_0)$, where $(x_0, y_0) \notin i(A)$. Let (a_n, b_n) be a sequence in A such that $(a_n, b_n) \leq (x_n, y_n)$ for all $n \in Z^+$. The convergence of the sequence (x_n, y_n) implies that the sequences (a_n) and (b_n) are both bounded above. Either of these sequences may fail to be bounded below, and this leads us to consider four cases.

Case 1. (a_n) and (b_n) are both bounded below. Then there is a convergent subsequence $(a_{n_k}, b_{n_k}) \to (a, b)$. Since A is closed, $(a, b) \in A$, and since R^2 is T_2-ordered, $(a, b) \leq (x_0, y_0)$, contrary to $(x_0, y_0) \notin i(A)$.

Case 2. (a_n) and (b_n) are both unbounded below. Then for some $n \in Z^+$, $(a_n, b_n) \leq (x_0, y_0)$, which again contradicts $(x_0, y_0) \notin i(A)$.

Case 3. (a_n) is bounded below, but (b_n) is not. In this case, there is no loss of generality in assuming that $a_n \to a$ and that (b_n) is an unbounded, decreasing sequence. Then there is $n_0 \in Z^+$ such that $b_n \leq y_0$, for all $n \geq n_0$. Also $a \leq x_0$ since R^1 is T_2-ordered, and for $n \geq n_0$ we must have $a \leq a_n$, for otherwise $(a_n, b_n) \leq (x_0, y_0)$ would again yield a contradiction. Thus from the sequence $(a_n)_{n \geq n_0}$ it is possible to obtain a decreasing subsequence $(a_{n_k}) \to a$, and the corresponding subsequence (b_{n_k}) is, of course, decreasing and unbounded. Now for any $j \in Z^+$, $(a_{n_j}, b_{n_j}) \leq (a_{n_j}, b_{n_1}) \leq (a_{n_1}, b_{n_1})$, and the convexity of A implies that $(a_{n_j}, b_{n_1}) \in A$ for all $j \in Z^+$. Thus, $(a_{n_j}, b_{n_1}) \to (a, b_{n_1})$ and $(a, b_{n_1}) \in A$ since A is closed. But $a \leq x_0$ and $b_{n_1} \leq y_0$ implies $(x_0, y_0) \in i(A)$, a contradiction.

Case 4. (b_n) is bounded below, but (a_n) is not. An argument similar to that of case 3 yields a contradiction. ∎

Proposition 3.2 R^n is not a c-space for $n \geq 3$.

Proof. Let $A = \{(m, -\frac{1}{m}, \frac{1}{m}, 0, \cdots, 0) \in R^n : m \in Z^+\}$. The elements of A are isolated in both the topological and order sense, and so A is a closed, convex subset of R^n. One can verify that $I(A) = i(A)$ and $D(A) = d(A) \cup B$, where $B = \{(z, 0, z, 0, \cdots, 0) \in R^n : z \leq 0\}$. Since $i(A) \cap B = \phi$, it follows that $A = I(A) \cap D(A) = A^\wedge$, and so A is a c-set. But $d(A) \neq D(A)$, and so R^n is not a c-space. ∎

Theorem 3.3 R^n is T_4-ordered for all $n \in Z^+$.

Proof. If $x = (x_1, \cdots, x_n) \in R^n$ and $x \leq y$, then $N(x, \epsilon) \subseteq d(N(y, \epsilon))$ and $N(y, \epsilon) \subseteq i(N(x, \epsilon))$; from this it easily follows that the increasing and decreasing hulls of open sets in R^n are open. If A is a closed increasing set and B a closed, decreasing set in R^n such that $A \cap B = \phi$, then for each $b \in B$ we may choose

r_b such that $A \cap N(b, r_b) = \phi$, and consequently $A \cap d(N(b, r_b)) = \phi$. Likewise, for each $a \in A$, there is r_a such that $B \cap i(N(a, r_a)) = \phi$. Let $U = \cap\{i(N(a, r_a/2)) : a \in A\}$ and $V = \cup\{d(N(b, r_b/2)) : b \in B\}$. Then U and V are disjoint open sets, the former increasing and the latter decreasing, which separate A and B. ∎

Theorem 3.4 The following statements are equivalent.

(a) R^n is a c-space.

(b) w_0R^n is T_1-ordered.

(c) w_0R^n is T_2-ordered.

(d) $w_0R^n = \beta_0R^n$.

(e) $n \leq 2$.

Proof. It is obvious that in R^1, the increasing or decreasing hull of any closed set is closed, and so R^1 is a c-space. Thus (a) ⇔ (c) follows by Theorem 3.1 and Proposition 3.2.

(a) ⇔ (b) follows from Theorems 2.7 and 2.8. By Theorems 1.6, 3.1, and 3.3, w_0R^n is T_2-ordered for $n \leq 2$, and by Theorem 1.6 and Proposition 3.2, w_0R^n is not T_2-ordered for $n \geq 3$; thus (c) ⇔ (e). Finally, (c) ⇔ (d) follows by Theorem 1.6 and Corollary 1.8. ∎

The Wallman ordered compactification of R^1 is the familiar two point compactification which is commonly called the "extended real line." In the case of R^2, this compactification, which is not so familiar, is described in the next example.

Example 3.5 R^2 is simultaneously homeomorphic and order isomorphic to the open square $S = \{(x_1, x_2) \in R^2 : -1 < x_1 < 1,\ -1 < x_2 < 1\}$. The closed square $\bar{S} = \{(x_1, x_2) \in R^2 : -1 \leq x_1 \leq 1,\ -1 \leq x_2 \leq 1\}$ can thus be regarded as a T_2-ordered compactification of R^2. The most convenient way to describe w_0R^2 (or, equivalently, β_0R^2) is to is to consider each boundary (i.e., compactification) point p of $\bar{S}$ to be replaced by the set $\mathcal{M}_p$ of all non convergent maximal c filters $\mathcal{H}$ on S which converge to p in $\bar{S}$. If $p = (1,1)$, then $\mathcal{M}_p$ consists of a single maximal c-filter which is the greatest element of w_0R^2. Similarly, the least element of w_0R^2 is the unique maximal c-filter in $\mathcal{M}_{(-1,-1)}$. If p is any boundary point of $\bar{S}$ other than $(1,1)$ or $(-1,-1)$, then $\mathcal{M}_p$ contains 2^γ distinct elements, where γ is the cardinality of the real line, including both a greatest and a least element. For instance, if $p = (1,0)$ the least element in $\mathcal{M}_p$ is a maximal c-filter in S which contains the positive x axis and converges in $\bar{S}$ to $(1,0)$; the greatest element is a maximal c-filter finer than the filter supremum of $\{I(\mathcal{F}) : \mathcal{F} \in \mathcal{M}_p\}$ which converges to $(1,0)$ in $\bar{S}$.

If p,q are two boundary points in $\bar{S}$ and $p \leq q$ in $\bar{S}$, then $\mathcal{G} \lesssim \mathcal{H}$ for all $\mathcal{G} \in \mathcal{M}_p$ and for all $\mathcal{H} \in \mathcal{M}_q$; furthermore if $\mathcal{G} \lesssim \mathcal{H}$ for some $\mathcal{G} \in \mathcal{M}_p$ and for some $\mathcal{H} \in \mathcal{M}_q$, then $p \leq q$ in $\bar{S}$. Similarly if $x \in S$ and p is a boundary point of $\bar{S}$, then $x \leq p$ in $\bar{S}$ iff $\mathring{x} \lesssim \mathcal{H}$ for some $\mathcal{H} \in \mathcal{M}_p$ (in which case $\mathring{x} \lesssim \mathcal{H}$ for all $\mathcal{H} \in \mathcal{M}_p$). ∎

The next two examples show that even for the simplest subspaces X of R^2, various pathologies can arise in $w_0 X$.

Example 3.6 Let X_1 be the closed square $\bar{S}$ (defined in Example 3.5) with the origin $(0,0)$ deleted, and with the topology and order inherited from R^2. Let $\mathcal{G}$ (respectively, $\mathcal{H}$) be the maximal c-filter on X_1 which contains the negative portion of the x-axis (respectively, y-axis) and converges to $(0,0)$ in $\bar{S}$. If $A = \{(x,0) : x < 0\}$ and $B = \{(0,y) : y < 0\}$, then A and B are c-sets in X_1. Since $B \subseteq D(A)$ but $B \cap d(A) = \phi, d(A) \neq D(A)$ and thus X_1 is not a c-space. Furthermore, it follows from Theorem 2.8 that $w_0 X_1$ is not T_1-ordered. Also, A and B cannot be separated by fundamental open sets, and consequently $w_0 X_1$ is not T_2. However, the argument used to prove Theorem 3.3 can be applied to show that X_1 is T_4-ordered. We thus have an example which, in contrast to Example 2.9, is normally ordered but not c-normally ordered, and for which the Wallman ordered compactification is neither T_1-ordered nor T_2.

It is easy to describe $w_0 X_1$. The "hole" at $(0,0)$ in X_1 is filled in $w_0 X_1$ by a set $\mathcal{M}_{(0,0)}$ of maximal c-filters on X_1 which converge to $(0,0)$ in $\bar{S}$. The filters $\mathcal{G}$ and $\mathcal{H}$ described above are minimal elements in $\mathcal{M}_{(0,0)}$; there are also two maximal elements in $\mathcal{M}_{(0,0)}$ which are maximal c-filters converging to $(0,0)$ in $\bar{S}$ along the positive x and y axes. The set of compactification points contains no greatest or least element and has cardinality 2^{γ}. ∎

Example 3.7 Let X_2 be the closed square $\bar{S}$ with the y axis deleted except for the origin; the topology and order are those inherited from R^2. One may show that this space is both T_4-ordered and c-normally ordered. However, X_2 is not a c-space, for if $A = \{(x, \frac{1}{2}) : x < 0\}$, then A is a c-set and $(0,0) \in D(A)$. Thus, $w_0 X_2$ is T_2 by Theorem 2.3, but $w_0 X_2$ is not T_1-ordered by Theorem 2.8. Without going into detail, we can partially describe $w_0 X_2$ by remarking that every "hole" in X_2 corresponding to a missing point on the y-axis is filled in $w_0 X_2$ by adding 2^{γ} compactification points including, in each case, a pair of minimal elements and a pair of maximal elements. ∎

For the sake of completeness, we should give an example of a space X for which $w_0 X$ is T_1-ordered but not T_2. This turns out to be quite easy. Let X be any T_2, completely regular toplogical space which is not normal, and let the order for X be equality. Then it is well known that w_0X is T_1 (and hence T_1-ordered) but not T_2.

Our final example does not pertain directly to the Wallman ordered compactification, but it does provide further insight into the nature of c-sets, which are crucial ingredients in the construction of this compactification. It shows that closed, convex subsets of R^3 need not be c-sets, and that the relation defined by "A is a c-set in B" is not transitive. We are grateful to Dr. Bettina Zoeller for providing this example as well as the related example used in the proof of Proposition 3.2.

Example 3.8 Let $K = \{(m, -\frac{1}{m}, \frac{1}{m}) : m \in Z^+\} \cup \{(-m, \frac{1}{m}, -\frac{1}{m}) : m \in Z^+\}$ be a subset of R^3; note that K is closed and convex. Let $L = I(K) \cap D(K)$; then L is the union of K with the x-axis, and consequently K is not a c-set. Furthermore, observe that K is a c-set in L (considered as a subspace of R^3) and L is a c-set in R^3, but K is not a c-set in R^3.

Such an example cannot be found in R^n for $n \leq 2$, since in these spaces every closed, convex set is a c-set.

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