How to Recognize a Parabola

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Parabolas have many interesting properties which were perhaps more well known in centuries past. Many of these properties hold only for parabolas, providing a characterization which can be used to recognize (theoretically, at least) a parabola. Here, we present a dozen characterizations of parabolas, involving tangent lines, areas, and the well-known reflective property. While some of these properties are widely known to hold for parabolas, the fact that they hold only for parabolas may be less well known. These remarkable properties can be verified using only elementary techniques of calculus, geometry, and differential equations.

A parabola is the set of points in the plane which are equidistant from a point F called the focus and a line l called the directrix. If the directrix is horizontal, then the parabola is the graph of a quadratic function $p(x) = \alpha x^2 + \beta x + \gamma$. We will not distinguish between a function and its graph. A *chord* of a function $f(x)$ is a line segment whose endpoints lie on the graph of the function. A chord of $f(x)$ is a segment of a *secant line* to $f(x)$. By the equation of a chord, we mean the equation of the corresponding secant line.

Our first characterization of parabolas involves the area between a function and a chord having horizontal extent h.

Theorem (How to Recognize a Parabola) 1. *Suppose $f(x)$ is a differentiable function and for all real numbers a and h with $h > 0$, $l(a, h, x)$ is the secant line determined by the two points $(a, f(a))$ and $(a + h, f(a + h))$ on the graph of $f(x)$, separated horizontally by h units. Then $f(x)$ is a parabola if and only if the signed area*

$$A(a, h) = \int_a^{a+h} l(a, h, x)\, dx - \int_a^{a+h} f(x)\, dx$$

between the line $l(a, h, x)$ and the function $f(x)$ over the interval $[a, a + h]$ is a nonzero function of h alone, not dependent on a.

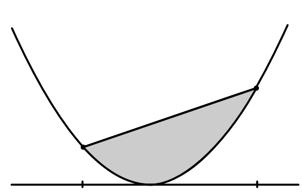

Figure 1. (Thm. 1) The shaded area depends on h alone, independent of the location a.

Proof. Archimedes knew that parabolas satisfy this property, illustrated in Figure 1. With techniques of calculus, it is easily verified that the area between $q(x) = x^2$ and the chord from (a, a^2) to $(a + h, (a + h)^2)$ is $h^3/6$, independent of a. Since any parabola $p(x) = \alpha x^2 + \beta x + \gamma$ is obtained from $q(x) = x^2$ by a translation,

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which does not change the area, and a vertical scaling/reflection by α, it follows that the signed area between $p(x) = \alpha x^2 + \beta x + \gamma$ and the chord from $(a, p(a))$ to $(a + h, p(a + h))$ is $\alpha h^3/6$. This is also proven without calculus in Swain and Dence [**16**].

Conversely, let $f(x)$, h, $l(a, h, x)$, and $A(a, h)$ be as in the statement of the theorem. The integral of $l(a, h, x)$ occurring in the definition of $A(a, h)$ is easily calculated and represents the area of a trapezoid under $l(a, h, x)$ and over $[a, a + h]$ if $f(a)$ and $f(a + h)$ are both positive, giving

$$A(a, h) = \frac{f(a) + f(a + h)}{2} h - \int_a^{a+h} f(x)\, dx.$$

Since this is constant relative to a, differentiating with respect to a and applying the Fundamental Theorem of Calculus for the derivative of the indefinite integral gives

$$0 = \frac{f'(a) + f'(a + h)}{2} h - f(a + h) + f(a),$$

or

$$\frac{f'(a + h) + f'(a)}{2} h = f(a + h) - f(a) \quad \text{for all } a, \text{ for all } h > 0. \tag{1}$$

Fixing $a = a_0$ and letting $x = a_0 + h$, the differential equation above becomes

$$\frac{x - a_0}{2} f'(x) - f(x) = \frac{-(x - a_0)}{2} f'(a_0) - f(a_0), \tag{2}$$

a linear first-order differential equation with nonconstant coefficients. In standard form the differential equation becomes

$$f'(x) - \frac{2}{x - a_0} f(x) = -f'(a_0) - \frac{2}{x - a_0} f(a_0). \tag{3}$$

Note that the coefficient of $f(x)$ has a discontinuity at $x = a_0$, as does the "forcing function" on the right side of the equation. The standard existence and uniqueness theorem guarantees a unique solution to any linear first-order differential equation through any specified initial condition, valid on the largest interval over which the coefficient functions and forcing functions are continuous. Thus, the differential equation (3) will have a unique solution f_1 on $(-\infty, a_0)$ through any specified initial values (x_0, y_0) with $x_0 < a_0$ and a unique solution f_2 on (a_0, ∞) through any specified initial values (x_1, y_1) with $a_0 < x_1$. Following standard procedures, (3) is solved by multiplying through by the integrating factor $\mu = e^{-\int 2/(x-a_0)\, dx} = (x - a_0)^{-2}$. This gives

$$(x - a_0)^{-2} f'(x) - 2(x - a_0)^{-3} f(x) = -(x - a_0)^{-2} f'(a_0) - 2(x - a_0)^{-3} f(a_0),$$

or

$$\frac{d}{dx}[(x - a_0)^{-2} f(x)] = -(x - a_0)^{-2} f'(a_0) - 2(x - a_0)^{-3} f(a_0).$$

Antidifferentiating both sides of the equation above gives

$$(x - a_0)^{-2} f(x) = (x - a_0)^{-1} f'(a_0) + (x - a_0)^{-2} f(a_0) + c,$$

so

$$f(x) = c(x - a_0)^2 + f'(a_0)(x - a_0) + f(a_0),$$

a quadratic function of x.

Thus, the general solution to the differential equation (3) will have the form

$$f(x) = \begin{cases} f_1(x) = c_1(x - a_0)^2 + f'(a_0)(x - a_0) + f(a_0) & \text{if } x < a_0 \\ f_2(x) = c_2(x - a_0)^2 + f'(a_0)(x - a_0) + f(a_0) & \text{if } x > a_0. \end{cases}$$

De Temple and Robertson [**4**] give a proof up to this point, arriving at a differential equation equivalent to (2), which they solve by different means. Repeating the argument above replacing a_0 by $a_1 > a_0$ will show that $f(x)$ must be parabolic on the interval $(-\infty, a_1)$ containing a_0, and thus f_1 and f_2 must be segments of a single parabola. ■

Dividing equation (1) by $h > 0$ provides the following corollary, illustrated in Figure 2.

Corollary (How to Recognize a Parabola) 2. *A nonlinear differentiable function $f(x)$ is a parabola if and only if the average of the slopes $f'(a)$ and $f'(a+h)$ of the lines tangent to $f(x)$ at the endpoints of every interval $[a, a+h]$ is equal to the slope of the chord connecting $(a, f(a))$ and $(a+h, f(a+h))$.* (*See Figure* 2.) *That is, a nonlinear differentiable function $f(x)$ is a parabola if and only if*

$$\frac{f'(a+h) + f'(a)}{2} = \frac{f(a+h) - f(a)}{h} \quad \textit{for all } a, \textit{ for all } h > 0. \tag{4}$$

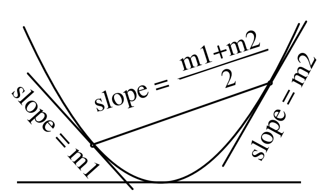

Figure 2. (Cor. 2) The slope of any chord is the average of the slopes of the tangents at the endpoints.

Proof. The proof of Theorem 1 showed that any function $f(x)$ with this property (4) is in fact a parabola, and conversely, it is easily verified that parabolas satisfy this property. ■

If a function $f(x)$ satisfies (4), our intuition would suggest that $f'(x)$ is changing at a constant rate over every interval $[a, a+h]$. That is, this suggests that the second derivative $f''(x)$ is constant, and thus that $f(x)$ is a parabola. The proof of Theorem 1 showed that this intuitive suggestion is correct.

We observe that this constant rate of change of the derivative $p'(x)$ of a parabola $p(x)$ is also illustrated by the fact that the point c guaranteed by the Mean Value Theorem applied to $p(x)$ over $[a-h, a+h]$ is the midpoint a of the interval. That is, if $p(x) = \alpha x^2 + \beta x + \gamma$, then

$$p'(a) = \frac{p(a+h) - p(a-h)}{2h} \quad \text{for all } a, \text{ for all } h > 0. \tag{5}$$

Again, this property characterizes parabolas, but we present some other preliminary results before presenting this as Theorem 4.

Theorem (How to Recognize a Parabola) 3. *Suppose f is a nonlinear function with a continuous derivative. Then $f(x)$ is a parabola if and only if the average of the slopes of the tangent lines at the endpoints of any interval equals the slope of the tangent line at the midpoint of the interval. That is, $f(x)$ is a parabola if and only if*

$$\frac{f'(a-h)+f'(a+h)}{2} = f'(a) \quad \textit{for all } a, \textit{ for all } h > 0. \tag{6}$$

See Figure 3.

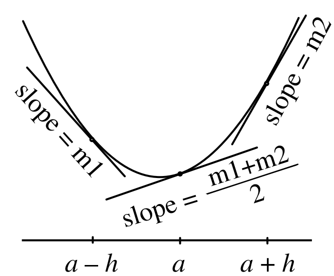

Figure 3. (Thm. 3) The slope at a is the average of the slopes at $a+h$ and $a-h$.

Proof. It is easy to verify that a quadratic function $f(x)$ satisfies (6), so suppose $f(x)$ has a continuous derivative and satisfies (6). Rewriting (6) as

$$\frac{f'(a)-f'(a-h)}{h} = \frac{f'(a+h)-f'(a)}{h},$$

we see that over any interval $[a-h, a+h]$ the point $(a, f'(a))$ on $f'(x)$ at the midpoint lies on the line $L(x)$ determined by the two points $(a-h, f'(a-h))$ and $(a+h, f'(a+h))$. Thus, the three points $(a+nh, f'(a+nh))$ $(n = 0, \pm 1)$ all lie on $L(x)$. Iterating this over the two halves of $[a-h, a+h]$, we find that the five points $(a+nh/2, f'(a+nh/2))$ $(n = 0, \pm 1, \pm 2)$ lie on $L(x)$. Further iterations show that $f'(x)$ coincides with $L(x)$ at all points $x = a + nh/2^k$ for $k \in \mathbb{N}$, $n \in \{0, \pm 1, \dots, \pm 2^k\}$. Thus, $f'(x)$ coincides with $L(x)$ on a dense subset of $[a-h, a+h]$, so by the continuity of $f'(x)$, it follows that $f'(x) = L(x)$ on $[a-h, a+h]$. Since $f'(x)$ is linear on any interval $[a-h, a+h]$ and $\mathbb{R}$ can be written as a union of overlapping intervals, it follows that $f'(x)$ is linear on $\mathbb{R}$, and thus $f(x)$ is a quadratic function. ■

This characterization of parabolas provides a nice way to recognize a linear function:

Lemma (How to Recognize a Line) 1. *Suppose f is a continuous function. Then $f(x)$ is a linear function if and only if the average value of $f(x)$ over every interval is the value of f at the midpoint of the interval. That is, $f(x)$ is a linear function if and only if*

$$\frac{\int_{a-h}^{a+h} f(x)dx}{2h} = f(a) \quad \textit{for all } a, \textit{ for all } h > 0. \tag{7}$$

Proof. If $f(x) = mx + b$ is a linear function, it is geometrically clear and easy to verify using calculus that (7) holds. Thus, suppose $f(x)$ satisfies (7). Since the left-hand side of (7) is a differentiable function of a, f is differentiable, and we have

$$f'(a) = \frac{f(a+h) - f(a-h)}{2h} \tag{8}$$
$$= \frac{1}{2}\left[\frac{f(a+h) - f(a)}{h} + \frac{f(a) - f(a-h)}{h}\right]$$
$$= \frac{f'(a+h/2) + f'(a-h/2)}{2}$$

by two applications of (8). Thus, the average of the slopes $f'(a-h/2)$ and $f'(a+h/2)$ at the endpoints of any interval $[a-h/2, a+h/2]$ is simply the value $f'(a)$ of the derivative at the midpoint of that interval. Since the right-hand side of (8) is a continuous function of a, f has a continuous derivative. Now by the proof of Theorem 3, it follows that $f(x)$ has the form $f(x) = \alpha x^2 + \beta x + \gamma$. We will show that $\alpha = 0$, so that $f(x)$ is in fact a linear function.

Since $f(x)$ satisfies (7), we have

$$\frac{\int_{a-h}^{a+h}(\alpha x^2 + \beta x + \gamma)dx}{2h} = \alpha a^2 + \beta a + \gamma.$$

Evaluating the integral above, we get

$$\alpha a^2 + \frac{\alpha h^2}{3} + \beta a + \gamma = \alpha a^2 + \beta a + \gamma \quad \text{for all } a, \text{ for all } h > 0.$$

Thus, $\alpha h^2/3 = 0$ for all $h > 0$, and consequently $\alpha = 0$, so $f(x) = \beta x + \gamma$ is a linear function, as needed. ■

Now we will show that the mean-value-at-the-midpoint property (5) satisfied by parabolas is in fact a characterization of parabolas: $f(x)$ is a parabola if and only if "centered chords are parallel," that is, chords horizontally centered at $x = a$ are parallel to the tangent line at $x = a$. See Figure 4.

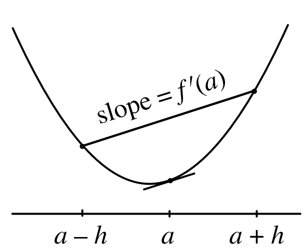

Figure 4. (Thm. 4) Chords horizontally centered at $x = a$ are parallel to the tangent at $x = a$.

Theorem (How to Recognize a Parabola) 4. *A nonlinear differentiable function $f(x)$ is a parabola if and only if over every interval $[a-h, a+h]$, the secant line $l(a, x)$ determined by $(a-h,\ f(a-h))$ and $(a+h,\ f(a+h))$ is parallel to the line tangent to f at the midpoint a of the interval. That is, f is a parabola if and only if*

$$f'(a) = \frac{f(a+h) - f(a-h)}{2h} \quad \textit{for all } a, \textit{ for all } h > 0. \tag{9}$$

Proof. A direct calculation or the argument preceding equation (5) shows that any parabola satisfies (9). Assume $f(x)$ satisfies (9). Since the right-hand side of (9) is a continuous function of a, we see that f' is continuous. Now since

$$\frac{f(a+h)-f(a-h)}{2h} = \frac{\int_{a-h}^{a+h} f'(x)dx}{2h} = f'(a) \quad \text{for all } a \text{, for all } h > 0,$$

by Lemma 1, $f'(x)$ is a linear function, so $f(x)$ is a parabola. ■

The characterization, as above, of parabolas as the functions for which the point c guaranteed by the Mean Value Theorem over any interval $[a-h, a+h]$ is the midpoint a has appeared in various forms. Haruki [**10**] and Aczel [**1**] show that if the slope of any chord of a function f over $[a-h, a+h]$ is a function of the midpoint a, then f is a parabola, even without assuming f is differentiable. Assuming f has derivatives of all order, Charlton [**3**] shows that if there exists a fixed number θ with $f(a+h)-f(a) = hf'(a+\theta h)$ for all a, h, then $\theta = 1/2$ and f is a parabola. Benyi, Szeptycki, and Van Vleck [**2**] also observed that the point c guaranteed by the Mean Value Theorem applied to f over $[a-h, a+h]$ is the point which maximizes the vertical distance between f and the secant line $l(a, x)$. This maximal vertical distance is the basis for another way to recognize a parabola. Before addressing that in Theorem 6, we present another characterization of parabolas geometrically similar to the previous one.

While the previous result showed that parabolas can be characterized by "centered chords being parallel," we now show that they may also be characterized by "parallel chords being centered," as illustrated in Figure 5. Smith [**14**] notes that if $p(x)$ is a parabola, the chords parallel to the tangent line at $x = a$ have midpoints which lie on the line $x = a$ for all a. The result below provides the converse as well.

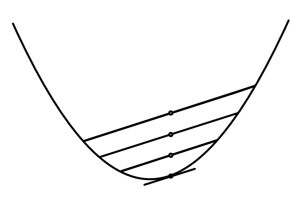

Figure 5. (Thm. 5) Chords parallel to the tangent at $x = a$ are horizontally centered at $x = a$.

Theorem (How to Recognize a Parabola) 5. *A differentiable function $f(x)$ is a parabola if and only if parallel chords have midpoints on a common vertical line $x = a$. Specifically, a differentiable function $f(x)$ is a parabola if and only if for any real numbers a_1 and a_2, and any positive real numbers h and k,*

$$\frac{f(a_1+h)-f(a_1-h)}{2h} = \frac{f(a_2+k)-f(a_2-k)}{2k} \quad \textit{implies } a_1 = a_2. \tag{10}$$

Proof. If $f(x)$ is a parabola and $(f(a_1+h)-f(a_1-h))/2h = (f(a_2+k)-f(a_2-k))/2k$, then by Theorem 4 we have $f'(a_1) = f'(a_2)$ and hence $a_1 = a_2$.

Now assume f is a differentiable function with the property (10) that any two parallel chords have midpoints on a common vertical line. First we observe that any line will intersect $f(x)$ in at most two points: If a line intersected $f(x)$ at three points

$x_1 < x_2 < x_3$, then the chords of f over the intervals $[x_1, x_2]$ and $[x_2, x_3]$ each have the same slope as the line, but do not have vertically aligned midpoints, contrary to the hypothesis.

Suppose a and $h > 0$ are given and let $l(x) = mx + b$ be the chord of f over the interval $[a-h, a+h]$. Let $c \in (a-h, a+h)$ be the point guaranteed by the Mean Value Theorem with $f'(c) = l'(c) = m$, and let $t(x) = mx + b_0$ be the line tangent to $f(x)$ at $x = c$. Now $l(x)$ and $t(x)$ are distinct parallel lines, and we assume that $l(x)$ lies above $t(x)$. The case of $l(x)$ below $t(x)$ is similar. Let $(b_n)_{n=1}^{\infty}$ be a decreasing sequence in (b_0, b) converging to b_0 and consider the parallel lines $l_n(x) = mx + b_n$. By the Intermediate Value Theorem (applied to $f(x) - t(x)$), $l_n(x)$ must intersect $f(x)$ at a point $v_n \in (a-h, c)$ and a point $w_n \in (c, a+h)$, and as noted above, at no other points. Now the chord of f over $[v_n, w_n]$ determined by $l_n(x)$ is parallel to the chord over $[a-h, a+h]$ and therefore has the same center a. Thus, $[v_n, w_n] = [a - h_n, a + h_n]$ for some $h_n > 0$, and the sequence $(h_n)_{n=1}^{\infty}$ is a decreasing sequence in $[0, h]$. As a decreasing sequence bounded below, $(h_n)_{n=1}^{\infty}$ must converge to a limit h_0, and since $c \in (a - h_n, a + h_n)$ for every natural number n, we have $c \in [a - h_0, a + h_0]$. It follows that $a = c$, for otherwise $a - h_0$, c, and $a + h_0$ would be three points on the intersection of $f(x)$ and the line $\lim_{n\to\infty} l_n(x) = t(x)$. Now the chord $l(x)$ is parallel to the line $t(x)$ tangent to f at $c = a$, which is the midpoint of the interval $[a-h, a+h]$, so Theorem 4 implies f is a parabola. ■

In his *Quadrature of the Parabola*, Archimedes states that parabolas satisfy the properties of Theorems 4 and 5, and cites the proofs from a Greek work *Elements of Conics* attributed to Aristaeus and Euclid ([**6**, pp. 50, 78]). Again, the converses were not mentioned.

Before characterizing parabolas by the maximal vertical distance from the function to a chord, we present a lemma.

Lemma 2. *Let g be a continuous function on $[a, b]$ with g'' defined on (a, b) and with $g(a) = g(b) = 0$. If $g''(x) < 0$ for $x \in (a, b)$, then $g(x) > 0$ for $x \in (a, b)$. If $g''(x) > 0$ for $x \in (a, b)$, then $g(x) < 0$ for $x \in (a, b)$.*

Proof. The second conclusion follows from the first by replacing g by $-g$, so we will prove only the first statement. Now $g''(x) < 0$ on (a, b) implies that g' is strictly decreasing on $[a, b]$. Since $g(a) = g(b) = 0$, it follows from the Mean Value Theorem that there exists $w \in (a, b)$ with $g'(w) = 0$. So, $g'(x) > 0$ for $x \in (a, w)$ and $g'(x) < 0$ for $x \in (w, b)$, and hence g is strictly increasing on $[a, w]$ and strictly decreasing on $[w, b]$. Since $g(a) = g(b) = 0$, it follows that $g(x) > 0$ for $x \in (a, b)$. ■

Applied to the difference $g(x)$ of a parabola $p(x)$ and a twice differentiable function $f(x)$ of the same concavity, this lemma simply tells us that if $f(x)$ is flatter than a parabola between two of its points of intersection with the parabola then, between these points, $f(x)$ is closer than the parabola to the secant line determined by the intersection points.

Theorem (How to Recognize a Parabola) 6. *Suppose f is a function for which f'' is continuous. For any $a \in \mathbb{R}$ and $h > 0$, let $l(a, h, x)$ be the secant line connecting $(a, f(a))$ and $(a+h, f(a+h))$, and let*

$$v(a, h) = \max_{x \in [a, a+h]} |l(a, h, x) - f(x)|$$

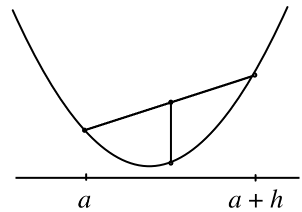

Figure 6. (Thm. 6) The maximum vertical distance from function to chord is a function of h alone, independent of a.

be the maximum vertical distance between $f(x)$ and $l(a, h, x)$. Then $f(x)$ is a parabola if and only if $v(a, h)$ is a nonzero function of h alone, independent of a. (See Figure 6.)

Proof. If $f(x) = p(x) = \alpha x^2 + \beta x + \gamma$ is a parabola, it is easy to verify that $v(a, h) = |\alpha| h^2/4$, independent of a.

Conversely, suppose $f(x)$ satisfies the hypotheses above but is not a parabola. Then $f''(x)$ is not constant, and by the continuity of f'', we can find disjoint intervals on which the values of f'' have the same sign but are separated by some constant k. That is, there exist $a_1, a_2, k \in \mathbb{R}$ and $h_0 > 0$ such that either $f''(x) > k > f''(y) > 0$ or $0 > f''(x) > k > f''(y)$ for all $x \in [a_1, a_1 + h_0]$ and all $y \in [a_2, a_2 + h_0]$. The cases are analogous, so we assume the former condition, in which f is concave up on the intervals in question. For $i = 1, 2$, let $p_i(x)$ be the parabola determined by $p_i''(x) = k$, $p_i(a_i) = f(a_i)$, and $p_i(a_i + h_0) = f(a_i + h_0)$. The condition on the second derivatives implies that $\alpha = k/2$ is the leading coefficients of both $p_1(x)$ and $p_2(x)$. Now applying Lemma 2 to $g_1(x) = f(x) - p_1(x)$, we have $g_1''(x) = f''(x) - p_1''(x) > 0$ over $(a_1, a_1 + h_0)$ so $f(x)$ lies below the parabola $p_1(x)$ on $(a_1, a_1 + h_0)$. Thus, the maximum vertical distance between f and $l(a_1, h_0, x)$ over $[a_1, a_1 + h_0]$ is greater than the maximum vertical distance $|\alpha| h_0^2/4$ between p_1 and $l(a_1, h_0, x)$ over the same interval. Consequently, if $v(a, h) = v(h)$ is a function of h alone, we have $v(h_0) > |\alpha| h_0^2/4$. But applying Lemma 2 to $g_2(x) = f(x) - p_2(x)$, we have $g_2''(x) = f''(x) - p_2''(x) < 0$ over $(a_2, a_2 + h_0)$ so f lies above p_2 on $(a_2, a_2 + h_0)$, and $v(h_0) < |\alpha| h_0^2/4$, a contradiction. Thus, if f is not a parabola, then $v(a, h)$ is not a function of h alone. ∎

We note that if f'' is continuous and $v(a, h) = v(h)$ is a nonzero function of h alone, then $v(h) = ch^2$ for some constant c, and $v(h)$ occurs at the midpoint $a + \frac{h}{2}$ of the interval $[a, a + h]$ for any a and any $h > 0$.

In Theorem 2 we saw a characterization of parabolas based on tangent lines at the ends of a chord. We now present another characterization of parabolas based on pairs of tangent lines. It has long been known (see [**14**]) that a parabola $p(x)$ has the property that the lines tangent to $p(x)$ at any points $x = a$ and $x = b$ intersect at the point $x = (a + b)/2$, as illustrated in Figure 7. The converse gives another way to recognize parabolas.

Theorem (How to Recognize a Parabola) 7. *A differentiable function $f(x)$ is a parabola if and only if the lines tangent to $f(x)$ at $x = a$ and $x = b$ intersect only at $x = (a + b)/2$ for all $a, b \in \mathbb{R}$.*

Proof. A direct argument shows that a parabola has the indicated property (see [**14**]). Assume $f(x)$ satisfies the hypotheses of the theorem. The lines $l_a(x) = f'(a)(x - a) + f(a)$ and $l_b(x) = f'(b)(x - b) + f(b)$ tangent to $f(x)$ at $x = a$ and $x = b$, respec-

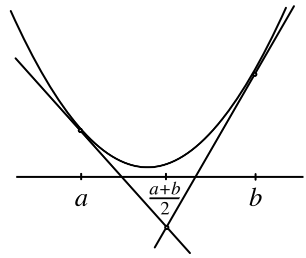

Figure 7. (Thm. 7) Tangent lines at $x = a$ and $x = b$ intersect at $x = \frac{a+b}{2}$.

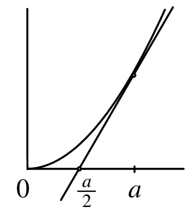

Figure 8. (Thm. 8) The tangent line at $x = a$ has x-intercept $\frac{a}{2}$.

tively, intersect when $f'(a)(x-a) + f(a) = f'(b)(x-b) + f(b)$. By the hypotheses, $x = (a+b)/2$ is the only solution to this equation, and substituting this value for x leads to

$$f'(a) + \frac{2}{b-a} f(a) = -f'(b) + \frac{2}{b-a} f(b).$$

Holding b fixed and letting a vary, this differential equation is equivalent to equation (3). As in the proof of Theorem 1, it follows that $f(a)$ is a parabola on $\mathbb{R}$. ■

One direction of our next result, illustrated in Figure 8, is obtained from Theorem 7 by taking $(b, f(b))$ to be the origin.

Theorem (How to Recognize a Parabola) 8. *Suppose $f(x)$ is a differentiable function with $f'(x) \neq 0$ for $x \neq 0$. Then $f|_{(-\infty,0]}(x)$ and $f|_{[0,\infty)}(x)$ are parabolic segments with vertices at the origin $(0, 0)$ if and only if the tangent line to $f(x)$ at $x = a$ has x-intercept $a/2$ for all $a \neq 0$.*

Fixing the x-axis as one tangent line simplifies the computation, but prevents us from shifting the argument to obtain a single parabola as the solution. (Observe that $f(x) = x^2$ for $x < 0$ and $f(x) = 2x^2$ for $x \geq 0$ is not a parabola, but satisfies the hypotheses of the theorem.)

Proof. With $f(x)$ as described, the x-intercept of the line tangent to $f(x)$ at $x = a$ is $x = a - f(a)/f'(a)$. Students will recognize this as the central step in Newton's method for approximating the zero of a differentiable function. Setting this x-intercept equal to $a/2$ leads to the differential equation $f'(a) - 2f(a)/a = 0$ for any $a \neq 0$. Multiplying by the integrating factor $\mu(a) = a^{-2}$ easily leads to $f(a) = ka^2$ on any interval in which the coefficient function $2/a$ is continuous, that is, on $(-\infty, 0)$ and on $(0, \infty)$. By continuity, $f(x)$ is a parabola on $(-\infty, 0]$ and on $[0, \infty)$, and $f(0) = 0$. The other direction follows from Theorem 7 with $b = 0$. ■

In his *Conica*, Appolonius shows that a parabola symmetric around the y-axis with vertex at the origin satisfies the equivalent property that the tangent at (x_0, y_0) has y-intercept $-y_0$ (see [**6**, p. 71]).

We now turn to one of the most widely known characterizing properties of the parabola, the reflective property illustrated in Figure 9a. While proofs that parabolic mirrors reflect rays parallel to the axis of symmetry into the focus are easily found, proofs that this property characterizes parabolas are less common. Drucker and

Locke [**8**] give a nice proof of this using concepts of geometry and calculus without differential equations. See also [**5**] and [**12**]. Both directions of this characterization were addressed by Diocles circa 200 B.C. in his treatise *On Burning Mirrors* [**17**].

Theorem (How to Recognize a Parabola) 9. *Suppose $f(x)$ is a differentiable function. Then $f(x)$ is a concave up parabola if and only if it has the property that all vertical rays coming downward from above $f(x)$ are reflected to a single point F.*

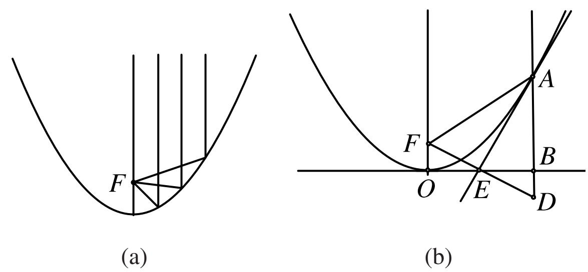

Figure 9. (Thm. 9) (a) Vertical rays are reflected to a single point F.

Proof. Suppose $f(x)$ is a concave up parabola and, without loss of generality, has vertex at the origin. Referring to Figure 9b, if a vertical ray intersects the function $f(x)$ at $A(a) = (a, f(a))$ and intersects the x-axis at $B(a) = (a, 0)$, then that vertical ray is reflected by the function $f(x)$ with an angle of reflection equal to the angle of incidence. Let O be the origin and let $F = (0, c)$ be the focus, so that $f(x) = x^2/(4c)$. Let $D(a)$ be the point on ray AB such that $AF = AD$, and let $E(a)$ be the intersection of $\overline{FD}$ and the line tangent to $f(x)$ at $x = a$. Since vertical angles are equal, the angle of incidence is $\angle EAB$. Then by the focus-directrix definition, $D(a)$ lies on the directrix, and hence $D(a) = (a, -c)$. Since $\overline{FD}$ has slope $-2c/a$ and $\overline{EA}$ has slope $f'(a) = a/(2c)$, we see that $\overline{FD} \perp \overline{EA}$. It follows that $\triangle EAF$ is congruent to $\triangle EAD$, so $\angle EAF = \angle EAD =$ the angle of incidence. Thus, if $f(x)$ is a parabola, any vertical ray is reflected at an angle which will pass through the focus F. This is the proof of [**14**] and [**18**].

Conversely, suppose vertical rays coming from above $f(x)$ are reflected to a point F, and without loss of generality, assume F is at the origin. Consider the right half of the curve—that is, the curve $f(x)$ for $x > 0$. Let $A = (x, f(x))$ be any point on this part of the curve. Let D be the point directly below A such that $AF = AD$. Then D has coordinates $(x, f(x) - \sqrt{x^2 + f(x)^2})$. Let E be the midpoint of FD, which has coordinates $(x/2, (f(x) - \sqrt{x^2 + f(x)^2})/2)$. Then $\triangle AFE$ is congruent to $\triangle ADE$, so $\overline{AE}$ bisects $\angle FAD$. Since the downward vertical ray through A reflects to F, $\overline{AE}$ must coincide with the tangent line at A, so we have

$$f'(x) = \frac{f(x) - (f(x) - \sqrt{x^2 + f(x)^2})/2}{x - x/2} = \frac{f(x) + \sqrt{x^2 + f(x)^2}}{x}. \tag{11}$$

Mueller and Thompson [**12**] indicate how to solve this differential equation. We proceed, avoiding the techniques of differential equations. We will show that the points D lie on a horizontal line $y = c$. Then from the focus-directrix definition, for $x > 0$, f would be a parabola with focus F and directrix $y = c$. The second coordinate of D is

$f(x) - \sqrt{x^2 + f(x)^2}$. To show that this is constant, consider its derivative:

$$\frac{d}{dx}\left(f(x) - \sqrt{x^2 + f(x)^2}\right) = f'(x) - \frac{x + f(x)f'(x)}{\sqrt{x^2 + f(x)^2}}$$
$$= \frac{f'(x)(\sqrt{x^2 + f(x)^2} - f(x)) - x}{\sqrt{x^2 + f(x)^2}}.$$

Now substituting the value of $f'(x)$ from equation (11) shows that the value of the expression above is zero. Thus, there is a constant c such that $f(x) - \sqrt{x^2 + f(x)^2} = c$. Noting that $c < 0$ and solving for $f(x)$, we get $f(x) = -x^2/(2c) + c/2$. Similar reasoning applies to the left half of the curve, and since the function is continuous at $x = 0$, the values of c for the two halves must agree, so $f(x)$ is a single parabola on $(-\infty, \infty)$. ■

We round out our dozen characterizations of parabolas with three which are proved elsewhere. The characterization below, given in [**13**], requires working in $\mathbb{R}^3$.

Theorem (How to Recognize a Parabola) 10. *A twice differentiable increasing function $f(x)$ on $[0, \infty)$ is a parabola on $[0, \infty)$ with vertex at the origin if and only if revolving that portion of f over the interval $[0, r]$ about the y-axis gives, for every $r > 0$, a bowl with exactly as much volume under it as inside it.* (*See Figure* 10.)

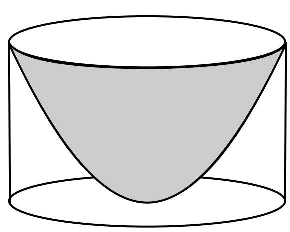

Figure 10. (Thm. 10) The volume under the bowl equals the volume in the bowl.

Another property of parabolas known to Archimedes is that the area between the secant line $l(a, h, x)$ and the parabola $p(x)$ as in Theorem 1 occupies two-thirds of the area of the circumscribing parallelogram having two vertical sides. See Figure 11. From Theorem 1, the area between the parabola $p(x) = \alpha x^2 + \beta x + \gamma$ and the chord over $[a, a + h]$ is $|\alpha|h^3/6$. The result then follows since the height (by Theorem 6) and width of the circumscribing parallelogram are $|\alpha|h^2/4$ and h, respectively. Archimedes' method of proof from his *Quadrature of the Parabola* is discussed in [**7**], [**15**], and [**16**], and a geometric proof from the 19th century is given in [**11**]. The proof

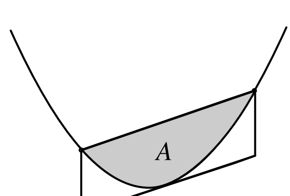

Figure 11. (Thm. 11) For any chord, the shaded area is $\frac{2}{3}$ that of the circumscribing parallelogram.

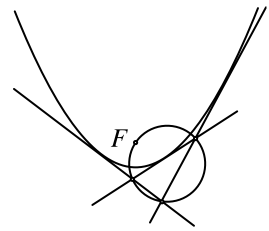

Figure 12. (Thm. 12) Points of intersection of any 3 tangents form a circle through a fixed point F.

of the converse is nontrivial and is found in [**2**]. We note that it requires the assumption that the function f be three times differentiable.

Theorem (How to Recognize a Parabola) 11. *If $f'''(x)$ exists and $f(x)$ is concave up, then $f(x)$ is a parabola if and only if the area between $f(x)$ and any chord from $(a, f(a))$ to $(a+h, f(a+h))$ is two-thirds the area of the circumscribing parallelogram having one pair of sides parallel to the y-axis.* (*See* [**2**].)

We conclude by citing the following geometric characterization from [**9**], illustrated in Figure 12.

Theorem (How to Recognize a Parabola) 12. *Suppose f is a differentiable function. Then f is a parabola if and only if any three lines tangent to f form a triangle whose vertices determine a circle which passes through a fixed point F, which is the focus of the parabola.* (*See* [**9**].)

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—Submitted by Carl C. Gaither, Killeen, TX