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ORDERED COMPACTIFICATIONS: CONNECTEDNESS AND REMAINDERS

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ABSTRACT. If $\beta_o X$ represents the Stone-Čech ordered compactification of an ordered topological space $(X, \tau, \leq)$ and A, B is a separation of X, we investigate when $\beta_o A, \beta_o B$ form a separation of $\beta_o X$. As an application, we show that there is no totally ordered topological space X which is homeomorphic and order isomorphic to its Stone-Čech ordered remainder $\beta_o X - X$.

1. Introduction

If X is a topological space, $C^*(X)$ represents the set of all continuous bounded real-valued functions on X. If $X \subseteq Y$ and every $f \in C^*(X)$ has a continuous bounded real-valued extension to Y, then we say X is C^*-embedded in Y. For each $f \in C^*(X)$, let I_f be a compact interval containing $f(X)$. Recall that the Stone-Čech compactification βX of a completely regular topological space X is the closure of $e(X)$, where $e : X \to \prod_{f \in C^*(X)} I_f$ is the evaluation map defined by $e(x) = \prod_{f \in C^*(X)} f(x)$. This construction of βX essentially includes a copy of each $f \in C^*(X)$ in the f^{th} coordinate of the product. Thus, every $f \in C^*(X)$ has a continuous bounded extension to βX, namely, $\pi_f \circ e$, so X is C^*-embedded in βX.

Indeed, the more general universal extension property of the Stone-Čech compactification is one of its most important properties: If Y is any compact T_2 topological space, and $f : X \to Y$ is continuous, then f has a continuous extension $\widehat{f} : \beta X \to Y$. In the language of category theory, this says that βX is the compact T_2 reflection of X. The related study of reflections and weak reflections has received more recent attention, as seen in [8, 9, 14, 15]. Further information on compactifications may be found in [3, 19]. Their relations to $C^*(X)$ are studied extensively in [7].

An ordered topological space $(X, \tau, \leq)$ is a topological space (X, τ) with a partial order $\leq$. The set of all continuous bounded increasing real-valued functions on

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$(X, \tau, \leq)$ will be denoted by $C^{*\uparrow}(X)$. An ordered topological space is T_2-ordered if the graph of the order is closed in $X \times X$ with the product topology, and is completely regular ordered if (a) for every $y \not\leq x$ in X, there exists $f \in C^{*\uparrow}(X)$ with $f(x) < f(y)$, and (b) for every closed set $F \subseteq X$ and every $a \notin F$, there exist $f, g \in C^{*\uparrow}(X)$ such that $f(a) = -g(a) = 1$ and $f(x) \wedge -g(x) \leq 0$ for all $x \in F$. A subset D of a poset X is *decreasing* if $d \in D$ and $x \leq d$ imply $x \in D$. Increasing sets are defined dually. A subset C of a poset is *convex* if $c, d \in C$ and $c \leq x \leq d$ imply $x \in C$. An ordered topological space X is *locally convex* if each point has a neighborhood base of convex sets. Every locally convex topology τ on a totally ordered space X is coarser than the interval topology. That is, all rays $(a, \rightarrow)$ and $(\leftarrow, b)$ and intervals (a, b) are always open in such topologies. See [6, 2].

A compact T_2-ordered space $(X^*, \tau^*, \leq^*)$ is an ordered compactification of the ordered topological space $(X, \tau, \leq)$ if (X, τ) is dense in (X^*, τ^*) (that is, (X^*, τ^*) is a compactification of (X, τ)) and $\leq^*$ is an extension of $\leq$. The Stone-Čech product construction applied to an ordered topological space $(X, \tau, \leq)$, using $C^{*\uparrow}(X)$ as the index set, produces the Stone-Čech ordered compactification $\beta_o X$, which is also called the Nachbin compactification. As expected, X is $C^{*\uparrow}$-embedded in $\beta_o X$ and every continuous increasing function f from X to an arbitrary compact T_2-ordered topological space Y has a continuous increasing extension $\widehat{f}$ to $\beta_o X$. A result of this extension property is that $\beta_o X$ is the largest ordered compactification of X, when the compactifications of X are ordered by $\alpha X \geq \gamma X$ if and only if there exists a continuous increasing function $f : \alpha X \rightarrow \gamma X$ which leaves the points of X fixed. According to this order, if $\alpha X \geq \gamma X$ as ordered compactifications, then either $\alpha X > \gamma X$ as topological compactifications or $\alpha X = \gamma X$ as topological compactifications and αX has a smaller order than γX. If X is a totally ordered topological space, then $\beta_o X$ is totally ordered and carries the interval topology. See [6, 16, 2, 12, 1].

A poset in which every nonempty subset has a supremum and an infimum is a *complete lattice*. If $(X, \tau, \leq)$ is a compact totally ordered topological space, then it is a complete lattice: If X is infinite and there exists a nonempty subset $S \subseteq X$ with no supremum, then $\mathcal{C} = \{(\leftarrow, x) : x \in S\} \cup \{(b, \rightarrow) : \text{b is an upper bound of } S\}$ is an open cover of X with no finite subcover.

The disjoint union of sets A and B will be denoted $A \mathbin{\mathring{\cup}} B$.

2. **Ordered Compactifications of Disconnected Spaces**

We mention the following old result, together with its ordered analog.

Proposition 1. *(a) A subspace S of a topological space (X, τ) is C^*-embedded in X if and only if $\beta S = cl_{\beta X} S$.*
(b) A subspace S of an ordered topological space $(X, \tau, \leq)$ is $C^{*\uparrow}$-embedded in X if and only if $\beta_o S = cl_{\beta_o X} S$, as ordered topological spaces.

Part (a) appears in [19, Proposition 1.48]. Part (b) follows by duplicating that proof and using the fact ([16, Theorem 6, p. 49]) that continuous increasing functions on a compact subspace of a normal space can be extended.

We will investigate when ordered versions of the following intuitive topological result hold. Recall that a *separation* of a topological space X is a pair of nonempty disjoint open subsets whose union is X, and X is connected if and only if it has no separation.

Proposition 2. *If A and B form a separation of a topological space X, then $\beta A = cl_{\beta X}A$, $\beta B = cl_{\beta X}B$, and these sets are disjoint, providing a separation of βX.*

Proof. Suppose A, B is a separation of topological space X. If $f \in C^*(A)$ has range $[-M, M]$, taking $\widehat{f}(B) = M + 1$ gives a continuous extension to $X = A \cup B$. Thus, A is C^*-embedded in X, so $\beta A = cl_{\beta X}A$ by Proposition 1 (a). Similarly, $\beta B = cl_{\beta X}B$. Now since A and B form a separation of X, there exists a continuous function $s : X \to \mathbb{R}$ with $s(A) = 0$ and $s(B) = 1$. If $\widehat{s}$ is the extension of s to βX, then $\widehat{s}(cl_{\beta X}A) = 0$ and $\widehat{s}(cl_{\beta X}B) = 1$, so $cl_{\beta X}A$ and $cl_{\beta X}B$ are disjoint. Since $\beta X = cl_{\beta X}X = cl_{\beta X}(A \cup B) = cl_{\beta X}A \cup cl_{\beta X}B$, we see that $cl_{\beta X}A$ and $cl_{\beta X}B$ form a separation of βX. □

To get the full strength of this result in the ordered setting, we would need to define an *ordered separation* of $(X, \tau, \leq)$ as a pair of subsets A, B of X such that there exists a continuous increasing surjection $s : X \to \{0, 1\}$, where $\{0, 1\}$ has the usual topology and order. However, this definition is too restrictive to be widely applicable. For example, since $s^{-1}(0) = s^{-1}((\leftarrow, 0.5))$ is a decreasing set and $s^{-1}(1) = s^{-1}((.5, \rightarrow))$ is an increasing set, an ordered separation must partition X into one increasing set and one decreasing set, not permitting any separation into nonconvex blocks A and B. So, we will not consider this strong form of "ordered separation", but only (topological) separations of ordered topological spaces. We start with totally ordered spaces. We will only consider totally ordered spaces X which are locally convex; by [6, Theorem 4.31], this is equivalent to X being completely regular ordered.

Proposition 3. *If A and B form a separation of a locally convex totally ordered topological space X and $cl_{\beta_o X}A \cap cl_{\beta_o X}B = \emptyset$, then $\beta_o A = cl_{\beta_o X}A$ and $\beta_o B = cl_{\beta_o X}B$, and thus these sets form a separation of $\beta_o X$.*

Proof. Under the hypotheses, $\beta_o X = \beta_o(A \cup B) = cl_{\beta_o X}(A \cup B) = cl_{\beta_o X}A \overset{\circ}{\cup} cl_{\beta_o X}B$. Note that $cl_{\beta_o X}A$ is locally convex: For any $x \in cl_{\beta_o X}A$, since $cl_{\beta_o X}A = \beta_o X - cl_{\beta_o X}B$ is open and $\beta_o X$ has a base of convex sets, there exists a convex neighborhood N of x with $N \subseteq cl_{\beta_o X}A$.

In [12], $\beta_o X$ is constructed as the set of all closed convex ultrafilters on X, and consequently, each closed convex ultrafilter on X has a limit in $\beta_o X$. Suppose $\mathcal{F}$ is an arbitrary closed convex ultrafilter on X. Now $\mathcal{F} \to x \in \beta_o X = cl_{\beta_o X} A \cup cl_{\beta_o X} B$, so $x \in cl_{\beta_o X} A$ or $x \in cl_{\beta_o X} B$. Assume $x \in cl_{\beta_o X} A$. Let N be a convex neighborhood of x contained in $cl_{\beta_o X} A$. Now N is an element of the filter $\mathcal{V}(x)$ of neighborhoods of x, and $\mathcal{F} \to x$ if and only if $\mathcal{V}(x) \subseteq \mathcal{F}$, so $N \in \mathcal{F}$. The restriction $\mathcal{F}|_N$ of $\mathcal{F}$ to N is a closed convex filter on X which contains $\mathcal{F}$, so $\mathcal{F}|_N = \mathcal{F}$ is a closed convex ultrafilter on A. Conversely, every closed convex ultrafilter $\mathcal{G}$ on $A \subseteq X$ is a closed convex ultrafilter on X. Thus, the set of closed convex ultrafilters on A equals the set of closed convex ultrafilters on X converging to a point of $cl_{\beta_o} A$. But, the set of closed convex ultrafilters on A is $\beta_o A$, and the set of closed convex ultrafilters on X converging to a point of $cl_{\beta_o X} A$ is the set of points of $\beta_o X$ having a limit point in $cl_{\beta_o X} A$, which is $cl_{\beta_o X} A$. Thus, $cl_{\beta_o X} A = \beta_o A$.

Similarly, $cl_{\beta_o X} B = \beta_o B$. □

If A and B are a separation of X, we note that the topological condition that $cl_{\beta X} A$ and $cl_{\beta X} B$ are disjoint, which was part of the conclusion of Proposition 2, had to be included in its ordered analog as part of the hypothesis in Proposition 3. The following example illustrates that this additional hypothesis may fail.

Example 4. Consider the subsets of $\mathbb{R}$ defined by $A = \{(4n, 4n+1) : n \in \mathbb{N}\}$ and $B = \{(4n+2, 4n+3) : n \in \mathbb{N}\}$, and $X = A \cup B$, with the usual topology and order. Now A and B clearly form a separation of X. Letting ∞_X be the maximum element of $\beta_o X$, we see that since A and B are "interlaced" by the order and $\beta_o X$ is locally convex, any neighborhood of ∞_X intersects both A and B. That is, $\infty_X \in cl_{\beta_o X} A \cap cl_{\beta_o X} B$.

A proof of Proposition 3 which avoids filters will follow from the following result.

Proposition 5. *If A, B is a separation of a locally convex totally ordered space X, then A and B are $C^{*\uparrow}$-embedded in X.*

Proof. Suppose $g : A \to [-M, M] \subseteq \mathbb{R}$ is continuous and increasing. Given $b \in B$, if $(\leftarrow, b] \cap A = \emptyset$, define $\widehat{g}(b) = -M - 1$. If $(\leftarrow, b] \cap A \neq \emptyset$, define $\widehat{g}(b) = \sup(g((\leftarrow, b] \cap A))$. This supremum must exist since $[-M, M] \subseteq \mathbb{R}$ is a complete lattice.

$\widehat{g}$ *is increasing:* The case $a_1 \leq a_2$ in A follows since $\widehat{g}|_A = g$. If $b_1 \leq b_2$ in B, then $(\leftarrow, b_1] \subseteq (\leftarrow, b_2]$, which implies $\widehat{g}(b_1) \leq \widehat{g}(b_2)$. If $a \leq b$ $(a \in A, b \in B)$, then $a \in (\leftarrow, b] \cap A$, so $\widehat{g}(a) \leq \widehat{g}(b)$. If $b \leq a$ $(a \in A, b \in B)$, then every point of $(\leftarrow, b] \cap A$ is less than a, so $\widehat{g}(b) \leq \widehat{g}(a)$.

$\widehat{g}$ *is continuous:* We will show that for every neighborhood U of $\widehat{g}(x)$, there exists a neighborhood N of x with $\widehat{g}(N) \subseteq U$. If $x = a \in A$, then $\widehat{g}(a) = g(a)$ and by the continuity of g, there exists a neighborhood N in A of a with $g(N) \subseteq U$. But since A is open in X, N is open in X and $\widehat{g}(N) = g(N) \subseteq U$. If $x = b \in B$, then since B is open and locally convex, there exists a convex neighborhood N of b in X with $N \subseteq B$. Then $\widehat{g}(N) = \widehat{g}(b) \subseteq U$, (since $z \in N$ implies $(\leftarrow, z] \cap A = (\leftarrow, b] \cap A$

because either $[z,b] \subseteq N \subseteq B$ or$[b,z] \subseteq N \subseteq B$). The proof that B is $C^{*\uparrow}$-embedded in X is dual. □

Now a direct application of Proposition 1 (b) and Proposition 5 provides an alternate proof of Proposition 3.

The example below shows that the result of Proposition 5 for totally ordered topological spaces does not hold for arbitrary partially ordered spaces.

Example 6. If X is a $T_{3.5}$-ordered locally convex partially ordered topological space and A, B is a separation of X, then A need not be $C^{*\uparrow}$-embedded in X. Let

$$\begin{aligned} A_1 &= [-1,0) \times \{-1\}, \\ A_2 &= (0,1] \times \{1\}, \\ A &= A_1 \cup A_2, \\ B &= [-1,1] \times \{0\}, \text{ and} \\ X &= A \cup B. \end{aligned}$$

Give X the subspace topology from $\mathbb{R}^2$ and the "up" order $(x,y) \leq (z,w)$ if and only if $x = z$ and $y \leq w$. Then X is $T_{3.5}$-ordered and locally convex, and A, B is a separation of X. Define $f : A \to \{0,1\}$ by $f(A_1) = 1$ and $f(A_2) = 0$. Since the order on A is discrete (i.e., equality), f is increasing and thus $f \in C^{*\uparrow}(A)$. Suppose $\widehat{f}$ is a continuous increasing extension of f to X. Now

$$\left(\frac{-1}{n}, -1\right) < \left(\frac{-1}{n}, 0\right) \Rightarrow 1 = \widehat{f}\left(\frac{-1}{n}, -1\right) \leq \widehat{f}\left(\frac{-1}{n}, 0\right) \text{ and}$$
$$\left(\frac{1}{n}, 0\right) < \left(\frac{1}{n}, 1\right) \Rightarrow \widehat{f}\left(\frac{1}{n}, 0\right) \leq \widehat{f}\left(\frac{1}{n}, 1\right) = 0.$$

Taking the limit as $n \to \infty$ gives the contradiction that $1 \leq \widehat{f}(0,0) \leq 0$, so f cannot be extended to a continuous increasing function on X.

Since A is not $C^{*\uparrow}$-embedded in X, Proposition 1 (b) implies that $\beta_o A \neq cl_{\beta_o X} A$. We will show this directly. Since $A = A_1 \cup A_2$ has the discrete order of equality and A_1 and A_2 form a separation of A, we have $\beta_o A = \beta A = \beta A_1 \mathring{\cup} \beta A_2$. That is, the Stone-Čech ordered compactification of A is the Stone-Čech compactification of A with equality as the order. On the other hand, $\beta X = \beta(A_1 \cup A_2 \cup B) = \beta A_1 \mathring{\cup} \beta A_2 \mathring{\cup} \beta B = \beta A_1 \mathring{\cup} \beta A_2 \mathring{\cup} B$.

We can make this into an ordered compactification by imposing the order from X together with $\alpha_1 < \alpha_2$ for all $\alpha_i \in \beta A_i - A_i$ $(i = 1,2)$. Since this is the smallest possible order that makes βX into an ordered compactification, we have that this ordered compactification is $\beta_o X$. Now $cl_{\beta_o X} A = \beta A_1 \mathring{\cup} \beta A_2$ with the order $\alpha_1 < \alpha_2$ for all $\alpha_i \in \beta A_i - A_i$ $(i = 1,2)$. Thus, as topological spaces, $\beta_o A = cl_{\beta_o X} A$, but they are not equal as ordered topological spaces since one carries equality as the order and the other has a nontrivial order.

3. **Remainders**

If αX is a compactification of X, the subspace $\alpha X - X$ is the associated *remainder*. Hussak [10], Jackson [11], and Stannett [17, 18] investigate *sequences of iterated remainders* of a completely regular space X, defined by $X_1 = \beta X - X$ and for $n \geq 1$, $X_{n+1} = \beta X_n - X_n$. They present the question of whether there are spaces X for which the sequence of iterated remainders is eventually constant, up to homeomorphism. For this to occur, we would have $X_n \approx X_{n+1} = \beta X_n - X_n$, where $\approx$ represents homeomorphism between topological spaces. Thus, a more general question is whether there are spaces $Y = X_n$ for which $\beta Y - Y \approx Y$.

We will use the results of the previous section to address the corresponding question for remainders of ordered compactifications of totally ordered topological spaces. We will show that if X is totally ordered and nonempty, then X is never homeomorphic and order isomorphic to $\beta_o X - X$. This also relates to a question of Kovar [13], who defines disjoint topological spaces Z and W to be *mutually compactificable by* K if there exists a compact space $K = Z \cup W$ having Z and W as subspaces, with every pair $(z, w) \in Z \times W$ separated by disjoint open sets in K. Our result shows that a totally ordered nonempty space X is never mutually compactificable with a copy of itself by $\beta_o X$. For ordered topological spaces, by $X \approx Y$ we mean X is homeomorphic to Y by a function which is also an order isomorphism.

Proposition 7. *If X is a locally convex totally ordered space and $X \approx \beta_o X - X$, then the largest and smallest elements of $\beta_o X$ are elements of X.*

Proof. Suppose the smallest element of $\beta_o X$ is $a \in \beta_o X$. Since a is the limit of points in X, there exists a decreasing net in X converging to $a \notin X$, and consequently, X has no smallest element. This contradicts $X \approx \beta_o X - X$. The dual argument shows that the largest element of $\beta_o X$ is an element of X. □

Suppose $f : X \to \beta_o X - X$ is a homeomorphism and order isomorphism. We may view f as a function from X to $\beta_o X$, and thus there is a unique continuous increasing extension $\widehat{f} : \beta_o X \to \beta_o X$. Our next result addresses this extension.

Proposition 8. *Suppose X is a locally convex totally ordered space. If $f : X \to \beta_o X - X$ is a homeomorphism and order isomorphism, then its continuous increasing extension $\widehat{f} : \beta_o X \to \beta_o X$ has no fixed points.*

Proof. Since f maps X to $\beta_o X - X$, clearly f fixes no points of X. By [5, Lemma 3.5.6] or [3, Theorem 1.6], if X is a dense subset of a Hausdorff space B, $\widehat{f} : B \to Y$ is continuous, and $\widehat{f}|_X$ is a homeomorphism into Y, then $\widehat{f}(B - X) \cap \widehat{f}(X) = \emptyset$. Thus $\widehat{f}(\beta_o X - X) \cap \widehat{f}(X) = \emptyset$, so for $c \in \beta_o X - X$, we have $\widehat{f}(c) \notin \widehat{f}(X) = f(X) = \beta_o X - X$, so $\widehat{f}(c) \in X$. □

Now consider the sets

$$\begin{aligned} C &= \{x \in X : f(x) > x\}, \\ D &= \{x \in X : f(x) < x\}, \\ \widehat{C} &= \{x \in \beta_o X : \widehat{f}(x) > x\}, \\ \text{and} \quad \widehat{D} &= \{x \in \beta_o X : \widehat{f}(x) < x\}. \end{aligned}$$

If a is the smallest element of $\beta_o X$, then $f(a) = \widehat{f}(a) > a$, so C and $\widehat{C}$ are nonempty. Similarly D and $\widehat{D}$ are nonempty. Since neither f nor $\widehat{f}$ have fixed points, C and D partition X while $\widehat{C}$ and $\widehat{D}$ partition $\beta_o X$.

Because $\widehat{f}$ has no fixed points, $\widehat{C} = \{x \in \beta_o X : \widehat{f}(x) \geq x\}$, and, as expected, this is a closed set: If $c \in cl_{\beta_o X}\widehat{C}$, then there is a net c_λ in $\widehat{C}$ converging to c. Now $\widehat{f}(c_\lambda) > c_\lambda$ for all λ, and taking the limit (using continuity of $\widehat{f}$, the T_2-ordered property of $\beta_o X$, and the fact that $\widehat{f}$ has no fixed points), we have $\widehat{f}(c) > c$, so $c \in \widehat{C}$. Thus, $\widehat{C} = cl_{\beta_o X}\widehat{C}$. In fact, $\widehat{C} = cl_{\beta_o X}C$: If there were a point $c \in \widehat{C} - cl_{\beta_o X}C$, then $c \in \beta_o X = cl_{\beta_o X}(C \cup D) = cl_{\beta_o X}C \cup cl_{\beta_o X}D$, so $c \in cl_{\beta_o X}D \subseteq \widehat{D}$, contrary to $c \in \widehat{C}$. Since X is a subspace of $\beta_o X$, it follows that C is closed in X. Similar arguments apply to D and $\widehat{D}$. We summarize these results.

Proposition 9. *With $C, D, \widehat{C}$ and $\widehat{D}$ as defined, C and D form a separation of X while $\widehat{C}$ and $\widehat{D}$ form a separation of $\beta_o X$.*

Now we give the main result of this section.

Theorem 10. *If X is a nonempty totally ordered topological space, then $X \not\approx \beta_o X - X$.*

Proof. Suppose to the contrary that $f : X \to \beta_o X - X \subseteq \beta_o X$ is a homeomorphism and an order isomorphism. Consider the unique continuous increasing extension $\widehat{f} : \beta_o X \to \beta_o X$, and let $C, D, \widehat{C}$, and $\widehat{D}$ be as defined above. Note that C and D satisfy the hypotheses of Proposition 3. Now $f : X \to \beta_o X - X$ may be viewed as $f : C \cup D \to (\beta_o C \cup \beta_o D) - (C \cup D) = (\beta_o C - C) \cup (\beta_o D - D)$. We now show that $f|_C$ is a homeomorphism and order isomorphism from C to $\beta_o C - C$. Every $z \in \beta_o C - C \subseteq \beta_o X - X$ is $f(x)$ for some $x \in X$. If $x \in D$ then $f(x) < x$, so $\widehat{f}(f(x)) < \widehat{f}(x) = f(x)$, so $z = f(x) \in \widehat{D}$, contrary to $z \in \beta_o C = \widehat{C}$. Thus, $f|_C : C \to \beta_o C - C$ is onto. It is one-to-one and continuous as a restriction, and similarly is an order isomorphism. Also, $(f|_C)^{-1} = f^{-1}|_{f(C)}$ is continuous. Thus, $C \approx \beta_o C - C$ by the function $f|_C$. But now Proposition 7 applies, giving that the largest element w of $\beta_o C$ is in C and satisfies $f(w) < w$. This contradicts $f(x) > x$ for all $x \in C$. □

We present a second proof of Theorem 10.

Proof. Suppose to the contrary that $f : X \to \beta_o X - X \subseteq \beta_o X$ is a homeomorphism and an order isomorphism. Consider the unique continuous increasing extension $\widehat{f} : \beta_o X \to \beta_o X$. Now $\beta_o X$ is a complete lattice. By the Tarski-Knaster fixed point theorem (see [4]), $\widehat{f}$ should have a fixed point, but this contradicts Proposition 9. □

Thus, there are no nonempty totally ordered topological spaces for which the sequence of iterated remainders is constant.

The example below shows that the sequence of iterated remainders may become periodic. We recall (see [12]) that for a totally ordered space, $\beta_o X$ consists of all the closed convex ultrafilters on X, and $\beta_o X$ carries the interval topology.

Example 11. Let $X_1 = \mathbb{Q}$, the set of rational numbers with the usual topology and order. For any irrational a, $\{(a, a+\epsilon] \cap \mathbb{Q} : \epsilon > 0\}$ and $\{[a-\epsilon, a) \cap \mathbb{Q} : \epsilon > 0\}$ are bases for nonconvergent closed convex ultrafilters on $\mathbb{Q}$, so, besides $\pm\infty$, $\beta_o\mathbb{Q} - \mathbb{Q}$ consists of two copies a^- and a^+ of each irrational a, with $x < a^- < a^+ < y$ for any $x, y \in \mathbb{Q}$ such that $x < a < y$. Now $X_2 = \beta_o X_1 - X_1$ has nonconvergent closed convex ultrafilters with bases $\{(r, r+\epsilon] : \epsilon > 0\}$ and $\{[r-\epsilon, r) : \epsilon > 0\}$ for each rational r, so $X_3 = \beta_o X_2 - X_2$ consists of two copies r^- and r^+ of each rational r, with $x < r^- < r^+ < y$ for any $x, y \in X_2$ with $x < r < y$. Since $[r^+, s^-] = (r^-, s^+)$ is open in X_3, we see that $X_3 \not\approx X_1 = \mathbb{Q}$ since no open set in X_1 has a least element. Now $\beta_o X_3$ will restore two copies of each irrational, as well as $\pm\infty$, so $X_4 = \beta_o X_3 - X_3 \approx X_2$, and the sequence of iterated remainders becomes periodic.

We note that if X_1 is the real line with the discrete topology and usual order, $\beta_o X_1$ adds $\pm\infty$ and two copies a^- and a^+ of each real number a. But $X_2 = \beta_o X_1 - X_1$ is compact, so $X_3 = \beta_o X_2 - X_2 = \emptyset$ and the sequence terminates.

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