COLLECTIONS OF MUTUALLY DISJOINT CONVEX SUBSETS OF A TOTALLY ORDERED SET

TYLER CLARK AND TOM RICHMOND

ABSTRACT. We present a combinatorial proof of an identity for $F_{2n+1}$ by counting the number of collections of mutually disjoint convex subsets of a totally ordered set of $n$ points. We discuss how the problem is motivated by counting certain topologies on finite sets.

Theorem. Given a totally ordered set $X$ of $n$ points, the number $C(n)$ of collections of mutually disjoint convex subsets of $X$ is given by

$$C(n) = 1 + \sum_{p=1}^{n} \sum_{j=1}^{p} \binom{n-p+j}{j} \binom{p-1}{j-1} = F_{2n+1}.$$

Proof. For any natural number $k$, let $\mathbb{k}$ denote the set $\{1, 2, \ldots, k\}$ with the usual total order $1 < 2 < \cdots < k$. Note that a convex subset of $\mathbb{k}$ is simply an interval in $\mathbb{k}$. Suppose $\mathcal{C}$ is a collection of mutually disjoint convex subsets of $X = \mathbb{n}$. We will call the members of $\mathcal{C}$ blocks. If $\mathcal{C}$ has $j$ blocks $(j = 0, \ldots, n)$ and $|\bigcup \mathcal{C}| = p$, these $p$ elements may be divided into $j$ convex blocks in $\binom{p-1}{j-1}$ ways by inserting $j-1$ dividers into the $p-1$ gaps between the $p$ points. Now we may totally order the $n-p$ points and $j$ blocks, by choosing which of the $n-p+j$ items will be blocks, in $\binom{n-p+j}{j}$ ways. Summing as $p$ goes from 1 to $n$ and as $j$ goes from 1 to $p$, and adding the one exceptional case corresponding to $j = 0$, we have

$$C(n) = 1 + \sum_{p=1}^{n} \sum_{j=1}^{p} \binom{n-p+j}{j} \binom{p-1}{j-1}.$$  \hspace{1cm} (1)

We may also find a recursive formula for $C(n)$. For any collection $\mathcal{C}$ of mutually disjoint convex subsets of $\mathbb{n}$, consider the point $n \in \mathbb{n}$. Now $n \notin \bigcup \mathcal{C}$ if and only if $\mathcal{C}$ is one of the $C(n-1)$ collections of mutually disjoint convex subsets of $n-1$. Furthermore, $n \in \{j+1, \ldots, n\} \in \mathcal{C}$ where, for now, $j \in \{1, 2, \ldots, n-1\}$, if and only if $\mathcal{C} \setminus \{\{j+1, \ldots, n\}\}$ is one of the $C(j)$ collections of mutually disjoint convex subsets of $j$. If $j = 0$, that is, if $n \in \{1, 2, \ldots, n\} \in \mathcal{C}$, then $\mathcal{C} = \{\mathbb{n}\}$ is the unique acceptable collection, and for this reason, we adopt the convention that $C(0) = 1$. Now summing over all cases $n \notin \bigcup \mathcal{C}$ and $n \in \{j+1, \ldots, n\} \in \mathcal{C}$ for $j = 0, 1, \ldots, n-1$, we have

$$C(n) = C(n-1) + \sum_{j=0}^{n-1} C(j).$$  \hspace{1cm} (2)

From either formula (1) or (2), we find the initial values of the sequence $\{C(n)\}_{n=0}^{\infty}$ to be $1, 2, 5, 13, 34, 89, \ldots$, which agree with the values of $F_{2n+1}$. Suppose $C(n) = F_{2n+1}$ for $n = 1, 2, \ldots, k-1$. From the recurrence formula (2) we have

FEBRUARY 2010
THE FIBONACCI QUARTERLY

\[ C(k) = F_{2k-1} + \sum_{j=0}^{k-1} F_{2j+1}. \]

Applying the identity \( \sum_{j=0}^{m} F_{2j+1} = F_{2m+2} \) (Identity \#2 in [2], noting their convention that \( f_n = F_{n+1} \)), we have \( C(k) = F_{2k-1} + F_{2k} = F_{2k+1} \). With the initial cases, this shows that \( C(n) = F_{2n+1} \) for all natural numbers \( n \). \( \square \)

The second half of the proof above, showing that \( C(n) = F_{2n+1} \), can also be accomplished using a tiling argument of Anderson and Lewis [1] which allows tiles of any length. Think of a convex subset of \( k \) as a white tile on a \( 1 \times k \) strip. Then a collection of mutually disjoint convex subsets of \( k \) may be represented by a tiling of a \( 1 \times k \) strip by white tiles of various lengths and red squares in any remaining gaps, and the number of such tilings is \( C(k) \). Having tiled a \( 1 \times k \) strip, we may obtain a suitable tiling of a \( 1 \times (k+1) \) strip either by appending a red square in the \( k+1 \)\(^{st} \) position (producing \( C(k) \) tilings), appending a white square in the \( k+1 \)\(^{st} \) position (producing \( C(k) \) tilings), or, if the tile covering the \( k \)\(^{th} \) slot is white, it may be expanded to cover the \( k+1 \)\(^{st} \) slot. To count these expansions easily, expand the tile covering the \( k \)\(^{th} \) slot, red or white, to cover the \( k+1 \)\(^{st} \) slot (in \( C(k) \) ways), then remove those \( C(k-1) \) ending in a red domino (and leaving a suitable tiling of a \( 1 \times (k-1) \) strip). Thus, \( C(k+1) = 3C(k) - C(k-1) \). This recurrence relation is satisfied by \( F_{2n+1} \) (see Identity \#7 in [2]), and since the initial terms agree, we conclude that \( C(n) = F_{2n+1} \) for all natural numbers \( n \). The authors are grateful to the referee for pointing out this tiling argument.

For a fixed \( p \), the second factors \((p-1)\) in the double sum of the theorem constitute the \((p-1)^{st} \) row of Pascal’s triangle, while the values of the first factors \((n-p+j)\) are a subset of the \((n-p)^{th} \) diagonal. Thus, the double-sum formula for \( F_{2n+1} - 1 \) can be viewed as the sum of dot products of vectors in Pascal’s triangle, as illustrated below for \( n = 4 \).

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 2 & 1 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1
\end{array}
\]

The sum of the dot products of the circled pairs of vectors is \( F_{2(4)+1} - 1 \).

Our motivation for this problem arose from counting certain finite topologies, as described below. If \( j \) is any point in a finite topological space, let \( N(j) \) be the intersection of all open sets containing \( j \).

**Corollary.** Let \( \mathcal{T} \) be the set of topologies \( \tau \) on \( n \) such that the basis \( \{N(j) : j \in \mathbb{N}\} \) consists of a collection \( \mathcal{C} \) of mutually disjoint convex subsets of \( \mathbb{N} \), or such a collection \( \mathcal{C} \) together with \( \mathbb{N} \). Then \( |\mathcal{T}| = F_{2n+1} - 1 \).

The corollary follows from the almost one-to-one correspondence between the topologies of \( \mathcal{T} \) and the collections \( \mathcal{C} \) counted by \( C(n) \), where for \( j \in \mathbb{N} \setminus \bigcup \mathcal{C} \), we take \( N(j) = \mathbb{N} \).
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However, the collection having no blocks generates the same topology—namely the indiscrete topology—as the collection having a single block containing all the points.

REFERENCES


MSC200: 11B39, 05A18, 54F05

DEPARTMENT OF MATHEMATICS, WESTERN KENTUCKY UNIVERSITY HONORS COLLEGE, 1906 COLLEGE HEIGHTS BLVD., BOWLING GREEN, KY 42101-1078

E-mail address: thomas.clark973@wku.edu

DEPARTMENT OF MATHEMATICS, WESTERN KENTUCKY UNIVERSITY, 1906 COLLEGE HEIGHTS BLVD., BOWLING GREEN, KY 42101-1078

E-mail address: tom.richmond@wku.edu