Finite-point order compactifications

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I

After the characterization of 1-point topological compactifications by Alexandroff in 1924, n-point topological compactifications by Magill [4] in 1965, and 1-point order compactifications by McCallion [5] in 1971, spaces that admit an n-point order compactification are characterized in Section 2. If $X^*$ and $X^{**}$ are finite-point order compactifications of $X$, $\sup \{X^*, X^{**}\}$ is given explicitly in terms of $X^*$ and $X^{**}$ in §3. In §4 it is shown that if an ordered topological space $X$ has an $m$-point and an $n$-point order compactification, then $X$ has a $k$-point order compactification for each integer $k$ between $m$ and $n$. The author is indebted to Professor Darrell C. Kent, who provided assistance and encouragement during the preparation of this paper.

Preliminaries. An ordered topological space, or simply an ordered space, is a triple $(X, \preceq, \theta)$ where $\preceq$ is a topology on $X$ and $\theta$ is the graph of a partial order on $X$. An order compactification of $(X, \preceq, \theta)$ is a pair $((X^*, \preceq^*, \theta^*), \kappa)$ such that $\kappa$ is an order embedding and $(X^*, \preceq^*, \theta^*)$ is a topological compactification of $(X, \preceq)$. We will make no distinction between $X$ and $\kappa(X)$. The property of a space being $T_{\alpha}$-ordered ($T_{\alpha^+}$-ordered) is equivalent to the space having a closed order (completely regular order) as defined in [7]. For other properties of ordered spaces, the reader is referred to [7] and [6]. All compactifications mentioned will be assumed to be $T_\alpha$, and all order compactifications will be assumed to be $T_\alpha$-ordered. If $(X^*, \preceq^*)$ and $(X^{**}, \preceq^{**})$ are order compactifications of ordered space $X$, then $X^* \succeq X^{**}$ means there exists a continuous increasing function $f : X^* \to X^{**}$ such that $f \circ \kappa = \kappa^{**}$. An ordered space $(X, \preceq, \theta)$ is said to have a convex topology if $\theta$ has a subbase of open monotone sets. It is known that if $X$ is $T_{\alpha^+}$-ordered, then $X$ has a convex topology.

Magill [4] has shown that a locally compact $T_2$ space $(X, \preceq)$ has an $n$-point compactification iff $X$ can be partitioned into $K \cup \{G_i\}_{i=1}^n$, where $K$ is compact, each $G_i$ is open, and $K \cup G_i$ is not compact for any $i$. Such a collection $\{G_i\}_{i=1}^n$ of disjoint open subsets of $X$ is called an $n$-star on $X$.

Throughout this paper, $i, j, k, m, n,$ and $p$ shall represent natural numbers, and $\mathcal{F}(x)$ shall represent the neighbourhood filter at $x$. Square brackets $[ ]$ are used to denote the filter generated by the base enclosed. The supremum $\mathcal{F} \lor \mathcal{G}$ of filters $\mathcal{F}$ and $\mathcal{G}$ exists iff $\mathcal{G} \subseteq \mathcal{F} \land \mathcal{G} \land \mathcal{F} \subseteq \mathcal{G} \lor \mathcal{G}$.

II

If $(X^*, \preceq^*)$ is an $n$-point topological compactification of a $T_{\alpha^+}$-ordered space $X$ with $X^* = X \cup \{\omega_i\}_{i=1}^n$, we want to find intrinsic conditions on $X$ which guarantee that the points $\{\omega_i\}_{i=1}^n$ can be ordered properly to make $X^*$ into an order compactification.
Suppose \((X, \tau, \theta)\) is a compact \(T_4\)-ordered space. For \(x \in X\), put \(\mathcal{W}(x) = \{\text{open increasing neighbourhoods of } x\}\), with \(\mathcal{W}(x)\) defined dually. If \(\omega_x \in X\) and \(\omega_x \notin \tau\), put \(\mathcal{F}_x = \{N \in \omega_x : \exists \mathcal{W}(x) \cap N \neq \emptyset\}\), and define \(\mathcal{F}_x\) dually. Then \(x \leq y\) if \(\mathcal{W}(x) \cap \mathcal{W}(y)\) exists, \(x \leq \omega_x\) if \(\mathcal{W}(x) \cap \mathcal{F}_x\) exists, \(\omega_x \leq x\) if \(\mathcal{W}(x) \cap \mathcal{W}(x)\) exists, and \(\omega_x \leq \omega_x\) if \(\mathcal{W}(x) \cap \mathcal{F}_x\) exists.

**2.1. Theorem.** A \(T_{1\frac{1}{2}}\)-ordered space \((X, \tau, \theta)\) has an \(n\)-point order compactification if and only if \((X, \tau)\) has an \(n\)-point topological compactification \(X^* = X \cup \{\omega_x\}_{x \in X}\) with corresponding \(n\)-star \(\{G_x\}_{x \in X}\), and

\[\forall x \in \{1, \ldots, n\}, \exists \text{ filter } \mathcal{F}_x \text{ on } X \text{ having a base of } \tau\text{-open } \theta\text{-increasing sets}\]

\[\forall x \in \{1, \ldots, n\}, \exists \text{ filter } \mathcal{F}_x \text{ on } X \text{ having a base of } \tau\text{-open } \theta\text{-decreasing sets}\]

\[\forall x \in X, \exists \text{ filter } \mathcal{W}(x) \text{ on } X \text{ having a base of } \tau\text{-open } \theta\text{-increasing sets}\]

\[\forall x \in X, \exists \text{ filter } \mathcal{W}(x) \text{ on } X \text{ having a base of } \tau\text{-open } \theta\text{-decreasing sets}\]

such that the collection of all of these filters satisfies:

1. \(\mathcal{F}_1 \cap \mathcal{F}_2 = \{N \subseteq X : (K \cup G_1) \cap N \text{ is compact}\}\) where \(K = X \setminus (\bigcup_{x \in G} G_x)\), and \(\mathcal{W}(x) \cap \mathcal{W}(x) = \mathcal{W}(x) = \{\tau\text{-neighbourhood filter at } x\}\).

2. \(\mathcal{W}(x) \cap \mathcal{W}(y)\) exists \(\iff (x, y) \in \theta\).

3. (a) \(\mathcal{F}_1 \cap \mathcal{F}_2\) exists \(\Rightarrow \mathcal{F}_1 \leq \mathcal{F}_2\) and \(\mathcal{F}_2 \leq \mathcal{F}_1\).

(b) \(\mathcal{F}_1 \cap \mathcal{W}(x)\) exists \(\Rightarrow \mathcal{W}(x) \leq \mathcal{F}_1\) and \(\mathcal{W}(x) \leq \mathcal{F}_2\).

(c) \(\mathcal{W}(x) \cap \mathcal{W}(x)\) exists \(\Rightarrow \mathcal{W}(x) \leq \mathcal{F}_1\) and \(\mathcal{W}(x) \leq \mathcal{F}_2\).

The proof of Theorem 2.1 will be presented after Lemma 2.2.

If \((X, \tau, \theta)\) is an ordered space with \(n\)-star \(\{G_x\}_{x \in X}\), a collection \(\{\mathcal{F}_x, \mathcal{W}(x)\}\), \(\mathcal{F}_x(x) = \{x \in \{1, \ldots, n\} : x \in X\}\) of filters on \(X\) having bases of \(\tau\)-open monotone sets as in Theorem 2.1 and satisfying the conditions 1–3 in Theorem 2.1 will be called an \(n\)-point order compactifying family on \(X\). Thus, \((X, \tau, \theta)\) has an \(n\)-point order compactification iff there exists an \(n\)-point order compactifying family on \(X\). A short proof of the lemma below is given in [8], lemma 2.10.

**2.2. Lemma.** Let \(X\) be a \(T_{3\frac{1}{2}}\)-ordered space with an \(n\)-point topological compactification \(X^* = X \cup \{\omega_x\}_{x \in X}\) corresponding to \(n\)-star \(\{G_x\}_{x \in X}\), and suppose \(\{\mathcal{F}_x, \mathcal{W}(x)\}\) is an \(n\)-point order compactifying family on \(X\). If \(x, y \in X\) and \(x < y\), then for each \(i \in \{1, \ldots, n\}\), either (a) \(\mathcal{W}(x) \cap \mathcal{F}_i\) does not exist, or (b) \(\mathcal{W}(y) \cap \mathcal{F}_i\) does not exist.

Proof of theorem 2.1. First suppose the \(T_{1\frac{1}{2}}\)-ordered space \((X, \tau, \theta)\) has an \(n\)-point order compactification \(X^* = X \cup \{\omega_x\}_{x \in X}\). For \(x \in X\), putting \(\mathcal{W}(x) = \{N \subseteq X : N \text{ is a } \tau\text{-open } \theta\text{-increasing neighbourhood of } x\}\), and putting \(\mathcal{F}_x = \{N \subseteq X : N \text{ is a } \tau\text{-open } \theta\text{-increasing neighbourhood of } \omega_x\}\), with \(\mathcal{W}(x)\) and \(\mathcal{F}_x\) defined dually, we can see that these filters form an \(n\)-point order compactifying family on \(X\).

Conversely, let \(X^* = X \cup \{\omega_x\}_{x \in X}\) be the \(n\)-point topological compactification of \(X\) whose existence is guaranteed by the existence of the \(n\)-point order compactifying family \(\{\mathcal{F}_x, \mathcal{W}(x)\}\). Define a relation \(\leq^*\) on \(X^*\) by putting \(x \leq^* y\) if \(\mathcal{W}(x) \cap \mathcal{W}(y)\) exists, \(x \leq^* \omega_x\) if \(\mathcal{W}(x) \cap \mathcal{F}_x\) exists, \(\omega_x \leq^* x\) if \(\mathcal{F}_x \cap \mathcal{W}(x)\) exists, and \(\omega_x \leq^* \omega_x\) if \(\mathcal{W}(x) \cap \mathcal{F}_x\) exists (for \(x, y \in X\) and \(x, y \in X^*\)). We claim that \(\leq^*\) is a partial order. Reflexivity is immediate. Antisymmetry: If \(x \leq^* y\) and \(y \leq^* x\), then \(x = y\). If \(x \leq^* \omega_x\) and \(\omega_x \leq^* x\), then \(x \leq \omega_x\) and \(\omega_x \leq x\), so \(\mathcal{F}_x \subseteq \mathcal{F}_x \subseteq \mathcal{F}_x \subseteq \mathcal{F}_x \subseteq \mathcal{F}_x\). Thus, \(\mathcal{F}_x \subseteq \mathcal{F}_x \subseteq \mathcal{F}_x \subseteq \mathcal{F}_x \subseteq \mathcal{F}_x\), and \(\mathcal{F}_x \subseteq \mathcal{F}_x \subseteq \mathcal{F}_x \subseteq \mathcal{F}_x \subseteq \mathcal{F}_x\). That \(X^*\) is a compact implies \(\mathcal{F}_x \subseteq \mathcal{F}_x\). A similar argument shows that \(\mathcal{F}_x \subseteq \mathcal{F}_x\), \(\mathcal{F}_x \subseteq \mathcal{F}_x\), \(\mathcal{F}_x \subseteq \mathcal{F}_x\), and \(\mathcal{F}_x \subseteq \mathcal{F}_x\), which is impossible.
since \( \mathcal{V}(x) \) has a base of compact sets and \( \mathcal{F} \) does not. Transitivity: Case 1: \( x \preceq y \), \( y \preceq z \). Then the existence of \( \mathcal{V}(x) \cap \mathcal{V}(y) \) and \( \mathcal{V}(y) \cap \mathcal{F} \) implies that both \( \mathcal{V}(x) \) and \( \mathcal{F} \) are contained in \( \mathcal{V}(y) \), so that \( \mathcal{V}(x) \cap \mathcal{F} \) exists, i.e. \( x \preceq z \).

Case 2: \( x \preceq y, y \preceq z \). By antisymmetry, \( x \neq y \). Since \( x \preceq \mathcal{F} \) and \( \mathcal{V}(y) \cap \mathcal{F} \) exists and \( \mathcal{V}(y) \preceq \mathcal{V}(y) \cap \mathcal{F} \), \( x \neq y \) would contradict Lemma 2.2.

Case 3: \( x \preceq y, x \preceq \mathcal{F} \). Then \( \mathcal{V}(x) \cap \mathcal{F} \) and \( \mathcal{V}(y) \cap \mathcal{F} \) exist, which implies, respectively, that \( \mathcal{V}(x) \subseteq \mathcal{F} \), and \( \mathcal{V}(y) \subseteq \mathcal{F} \), so that \( \mathcal{V}(x) \cap \mathcal{V}(y) \subseteq \mathcal{F} \) exists, i.e. \( x \preceq z \). The remaining cases are similar. Thus \( \preceq \) is a partial order.

We will denote the graph of \( \preceq \) by \( \theta^* \). From the definition of \( \preceq \) and condition 2 in the definition of order compactifying family, it is immediate that the order \( \theta^* \), when restricted to \( X \), agrees with the original order \( \theta \). All that remains is to show that \( (X^*, \tau^*, \theta^*) \) is \( T_2 \)-ordered. Suppose \( \alpha, \beta \in X^* \) and \( \alpha \preceq \beta \). To eliminate some repetitive cases, define \( \mathcal{V}_\alpha = \mathcal{V}(\alpha) \) if \( \alpha \in X^* \cap X \) and \( \mathcal{V}_\alpha = \mathcal{F} \) if \( \alpha = \omega \), with \( \mathcal{V}_\alpha \) and \( \mathcal{V}_\beta \) defined analogously. Now \( \alpha \preceq \beta \Rightarrow \mathcal{V}_\alpha \cap \mathcal{V}_\beta \) does not exist \( \Rightarrow \exists \) increasing \( \alpha' \in \mathcal{V}_\alpha \), decreasing \( \beta' \in \mathcal{V}_\beta \), such that \( N_\alpha \cap N_\beta = \emptyset \). Observe that \( \mathcal{N}(\alpha) \cup \beta' \) is a \( \theta^* \)-increasing neighbourhood of \( \alpha \) and \( \mathcal{N}_\beta \cup \beta' \) is a \( \theta^* \)-decreasing neighbourhood of \( \beta \) (monotone hulls taken in \( X^* \)). These neighbourhoods are disjoint as required, for

\[
\mathcal{N}(\alpha) \cup \beta' \cap \mathcal{N}_\beta \cup \beta' = [\mathcal{N}(\alpha) \cap \mathcal{N}_\beta] \cup [\mathcal{N}(\alpha) \cap \mathcal{N}_\beta] \cup [\mathcal{N}(\alpha) \cap \mathcal{N}_\beta] \cup [\mathcal{N}(\alpha) \cap \mathcal{N}_\beta],
\]

and it is easily shown that each of these last four bracketed terms must be empty.

Thus \( X^* \) is \( T_2 \)-ordered. \( \square \)

We now characterize ordered spaces for which every order compactification is a finite-point compactification. For \( A, B \subseteq X \), we will say \( A \) and \( B \) are monotonically separated if there exists a monotone continuous function \( f: X \to \mathbb{R} \) such that \( f(A) = 0 \) and \( f(B) = 1 \). \( \beta^*_X \) denotes the largest order compactification of \( X \), known as the Stone-Čech \( \beta \), or the Nachbin order compactification of \( X \).

2.3. Theorem. For a \( T_1 \)-ordered space \( X \), the following are equivalent:

(a) \( \beta^*_X \) has \( n \) points.

(b) \( n \) is the least cardinal number for which \( X \) has an n-point order compactification.

(c) There exist \( n \) closed, non-compact sets in \( X \) which are monotonically separated in pairs, but no such collection of \( n + 1 \) sets.

Proof. Clearly (a) and (b) are equivalent. After observing that if \( a_1, \ldots, a_n \) are elements of a compact \( T_1 \)-ordered space, then there exist compact neighbourhoods \( U_i \), of \( a_i \), \( i = 1, \ldots, n \) which are monotonically separated in pairs, the equivalence of (a) and (c) is a direct generalization of Firby's proof of the corresponding topological result ([3], theorem 22). \( \square \)

Note that the embedding of \( X \) into the order compactification \( X^* \) generated from the order compactifying family is \( \text{id}_X \). If the order on a space is discrete, an n-point order compactifying family can be obtained by taking \( \mathcal{F} = \mathcal{F}_1 = \mathcal{F}_2 \) and \( \mathcal{V}(x) = \mathcal{V}(x) \cap \mathcal{F}(x) \), and the results of 2.I reduce to Magill's results on n-point topological compactifications. Also, \( \omega_0 \) is minimal in \( X^* \) if we may take the filter \( \mathcal{F}_1 \) from the corresponding order compactifying family to be equal to \( \mathcal{F}_1 \).
Blatter [1] has shown that every non-empty set of $T_o$-ordered compactifications of a $T_o$-ordered space has a supremum (unique up to equivalence) with respect to the usual ordering. Here we give an explicit description of the supremum of two finite-point order compactifications of $X$. We say two $n$-stars on $X$ are equivalent if they generate equivalent compactifications of $X$. Recall that for an $n$-star $(G_i)_{i=1}^n$, $G_i$ is defined to be $(N \subseteq X : (K \cup G_i) \cap N$ is compact) where $K = X \backslash \bigcup_{i=1}^n G_i$. We define two stars $(G_i)_{i=1}^n$ and $(H_i)_{i=1}^n$ to be a reduced pair if $\bigcup_{i=1}^n G_i = \bigcup_{i=1}^n H_i$ and $G_i \cap H_i = \varnothing$ iff $\mathcal{G}_i \cup \mathcal{H}_i$ exists.

3-1. Proposition. Given any $n$-star $(G_i)_{i=1}^n$ on $X$ and any $m$-star $(H_i)_{i=1}^n$ on $X$, there exists a reduced pair of stars $(\hat{G}_i)_{i=1}^n$ and $(\hat{H}_i)_{i=1}^n$ on $X$ such that $(\hat{G}_i)_{i=1}^n$ is equivalent to $(G_i)_{i=1}^n$ and $(\hat{H}_i)_{i=1}^n$ is equivalent to $(H_i)_{i=1}^n$.

Proof. Let $K_1 = X \backslash \bigcup_{i=1}^n G_i$ and $K_2 = X \backslash \bigcup_{i=1}^n H_i$. If $K_1 \neq K_2$, put $K = K_1 \cup K_2$, $\hat{G}_i = G_i \cap K$, and $\hat{H}_i = H_i \cap K$. If $\hat{G}_1 = \emptyset$, then $G_1 \subseteq K_1$ and $\hat{G}_i$ is compact. But $X \backslash (K_1 \cup G_i)$ is open, so that $\text{cl}(G_i) \subseteq K_1 \cup G_i$, contrary to $K_1 \cup G_i = K_2 \cup \text{cl}(G_i)$ being non-compact. Thus, each $\hat{G}_i$ is non-empty. Observe that $K$ is compact, $X = K \cup \bigcup_{i=1}^n G_i$, and $\{\hat{G}_i\}_{i=1}^n$ is a collection of disjoint open sets. Since $K \cap G_i$ is not compact but $K_1 \cup G_i \cup \{\omega_i\}$ is (where $\omega_i$ is the compactification point corresponding to $G_i$ in Magill's compactification), there is an ultrafilter $\mathcal{F} \rightarrow \omega_i$ with $K \cup G_i \in \mathcal{F}$, so $\hat{G}_i \subseteq K \cup G_i$ is not compact. Thus, $\{\hat{G}_i\}_{i=1}^n$ is an $n$-star. Similarly, $\{\hat{H}_i\}_{i=1}^n$ is an $n$-star, and $X \backslash (\bigcup_{i=1}^n \hat{G}_i) = X \backslash (\bigcup_{i=1}^n \hat{H}_i) = K$. The $n$-stars $(\hat{G}_i)_{i=1}^n$ and $(\hat{H}_i)_{i=1}^n$ are equivalent if and only if $([N \subseteq G_i : (K \cup G_i) \cap N \text{ is compact}]) = ([N \subseteq \hat{G}_i : (K \cup \hat{G}_i) \cap N \text{ is compact}])$, and this is easily verified.

Now assume that the original stars $(G_i)_{i=1}^n$ and $(H_i)_{i=1}^n$ satisfy $\bigcup_{i=1}^n G_i = \bigcup_{i=1}^n H_i$. We claim that if $G_i$ is replaced by any $\tilde{G}_i \subseteq G_i$ such that $(K \cup G_i) \hat{G}_i$ is compact, the resulting collection is an equivalent $n$-star. Since $(K \cup G_i) \hat{G}_i$ is compact, $(K \cup G_i) \hat{G}_i = (K \cup G_i) \hat{G}_i$ is closed. Now $\hat{G}_i$, being the complement of this set, is open. Clearly $\hat{G}_i$ is disjoint from $G_i$ for $j \neq i$. If $\hat{G}_i$ were compact, then $(K \cup G_j) \cup (K \cup G_i) \hat{G}_i$ would be compact. But this union is $K \cup G_i$, which is not compact, so $K \cup G_i$ is not compact. Thus, replacing $G_i$ by $\tilde{G}_i$ does give a second $n$-star. Since $G_i$ and $\tilde{G}_i$ are both punctured neighbourhoods of $\omega_i$, the two $n$-stars are equivalent.

Now if $\mathcal{F}_i \cup \mathcal{F}_j$ does not exist, then $\exists \tilde{G}_i \in \mathcal{F}_i, \tilde{H}_i \in \mathcal{F}_j$ such that $\tilde{G}_i \cap \tilde{H}_j = \emptyset$. Replacing $G_i$ and $H_i$ by $\tilde{G}_i$ and $\tilde{H}_i$, respectively, gives an equivalent pair of stars satisfying the condition that $\mathcal{F}_i \cup \mathcal{F}_j$ exists iff $\tilde{G}_i \cap \tilde{H}_j = \emptyset$.

It is easily seen that if $(G_i)_{i=1}^n$ and $(H_i)_{i=1}^n$ are a reduced pair of stars on $X$, then $\mathcal{C} = (G_i \cap H_j : G_i \cap H_j = \emptyset)$ is a $p$-star on $X$ where $p$, the cardinality of $\mathcal{C}$, satisfies max$\{m, n\} \leq p \leq mn$.

3-2. Theorem. Suppose $X$ has an $n$-point order compactifying family $(\mathcal{F}_i, \mathcal{F}_j, \mathcal{U}_i(x), \mathcal{U}_j(x))$ and $m$-point order compactifying family $(\mathcal{H}_i, \mathcal{H}_j, \mathcal{V}_i(x), \mathcal{V}_j(x))$ associated with an $n$-star $(G_i)_{i=1}^n$ and $m$-star $(H_i)_{i=1}^n$, where $(G_i)_{i=1}^n$ and $(H_i)_{i=1}^n$ form a reduced pair. Then the collection

$$\mathcal{C} \equiv \{\mathcal{F}_i \cup \mathcal{F}_j, \mathcal{F}_i \cup \mathcal{F}_j, \mathcal{U}_i(x), \mathcal{U}_j(x), \mathcal{V}_i(x), \mathcal{V}_j(x) : \mathcal{F}_i \cup \mathcal{F}_j \text{ exists}\}$$

is a $p$-point order compactifying family on $X$, with max$\{m, n\} \leq p \leq mn$. 
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Proof. Since \( \mathcal{C} \) has \( p \) elements, let us name them \( \{d_i\}_{i=1}^p \). If \( J = G \cap H \), define \( \mathcal{F}_k = \mathcal{F}_k \vee \mathcal{F}_k \), \( \mathcal{F}_k = \mathcal{F}_k \vee \mathcal{F}_k \), \( \mathcal{W}^*(x) = \mathcal{W}^*(x) \vee \mathcal{W}^*(x) \), \( \mathcal{W}^*(y) = \mathcal{W}^*(y) \vee \mathcal{W}^*(y) \), with \( \mathcal{F}_k \) and \( \mathcal{W}^*(x) \) defined dually. Observe that \( \mathcal{W}^*(x) \vee \mathcal{W}^*(y) = \mathcal{W}^*(x) \vee \mathcal{W}^*(y) \) exists if \( x \leq y \). Thus, we must only show

1. \( \mathcal{F}_k = \mathcal{F}_k \vee \mathcal{F}_k \frac{\{N \in X: [N \cup (G \cap H) \setminus N] \text{ is compact}\}}{\} \}
2. \( \mathcal{F}_k \vee \mathcal{F}_k \exists \mathcal{F}_k \leq \mathcal{F}_k \) and \( \mathcal{F}_k \leq \mathcal{F}_k \)
3. \( \mathcal{F}_k \vee \mathcal{W}^*(x) \exists \mathcal{F}_k \leq \mathcal{W}^*(x) \) and \( \mathcal{W}^*(x) \leq \mathcal{F}_k \)
4. \( \mathcal{W}^*(x) \vee \mathcal{F}_k \exists \mathcal{W}^*(x) \leq \mathcal{F}_k \) and \( \mathcal{F}_k \leq \mathcal{W}^*(x) \)

For (1), suppose first that \( N \in \mathcal{F}_k \) is a basic set, i.e., \( N = N \cap N \) where \( (K \cup G) \cap N \) and \( (K \cup H) \cap N \) are compact. Because \( K \cap H \) and \( K \cap G \) are closed,

\[ \{K \cup G \cap H) \setminus N = \{[K \cup G] \cap N \cap (K \cup H)] \cup ([K \cup H \cap N)] \setminus (K \cup G) \} \]

is compact, and thus \( \mathcal{F}_k \leq \mathcal{F}_k \). Conversely, suppose \( \{K \cup G \cap H) \setminus N \) is compact. Put \( N = N \cap N \) and \( N = N \cap N \). Now \( N \cap N = N \) and \( (K \cup G) \cap N = (K \cup G) \cap N \), which is compact. Similarly, \( (K \cup H) \cap N \) is compact. This completes the proof of (1).

For (2), we must show that \( \mathcal{F}_k \vee \mathcal{F}_k \exists \mathcal{F}_k \leq \mathcal{F}_k \) and \( \mathcal{F}_k \leq \mathcal{F}_k \). Rename indices (possibly introducing some repetition in the collection \( \{G_k\}_{k=1}^p \), since \( p \geq n \)) so that \( \mathcal{F}_k \leq \mathcal{F}_k \), \( \mathcal{F}_k \leq \mathcal{F}_k \) and \( \mathcal{F}_k \leq \mathcal{F}_k \). Now \( \mathcal{F}_k \vee \mathcal{F}_k \exists \mathcal{F}_k \leq \mathcal{F}_k \) and \( \mathcal{F}_k \leq \mathcal{F}_k \), where \( \mathcal{F}_k \leq \mathcal{F}_k \), \( \mathcal{F}_k \leq \mathcal{F}_k \), \( \mathcal{F}_k \leq \mathcal{F}_k \), \( \mathcal{F}_k \leq \mathcal{F}_k \). This completes the proof of (2). Similar arguments work for (3) and (4).

In \( X \), the order compactification generated from the order compactifying family in Theorem 32, define \( \mu_{ij} \) to be the compactification point having punctured neighbourhood filter \( \mathcal{F}_i \vee \mathcal{F}_j \). Thus, \( X^* = X \cup \{\mu_{ij}: \mathcal{F}_i \vee \mathcal{F}_j \exists \mathcal{F}_k \} \), and the compactification point \( \mu_{ij} \) in \( X^* \) is associated with the compactification points \( \omega_i \) in \( X^* \) and \( \omega_j \) in \( X^* \). With this notation, we will see in Theorem 34 that \( X^* = \sup\{X^*, X^{**}\} \).

33. Lemma. If \( (X^*, \tau^*, \theta^*) \) and \( (X^{**}, \tau^{**}, \theta^{**}) \) are finite-point order compactifications of \( (X, \tau, \theta) \), and if \( (X, \tau^*, \theta^*) \) is the order compactification generated from \( X^* \) and \( X^{**} \) as in Theorem 32, with \( X^* = X \cup \{\mu_{ij}: \mathcal{F}_i \vee \mathcal{F}_j \exists \mathcal{F}_k \} \), then

\[ x < \mu_{ij} \Leftrightarrow x < \omega_i \text{ and } x < \omega_j \]
\[ \mu_{ij} < x < \omega_i \text{ and } \mu_{ij} < \omega_j \]
\[ \mu_{ij} < \mu_{kl} \Leftrightarrow \omega_i < \omega_k \text{ and } \mu_{ij} < \omega_l \]
\[ \mu_{ij} < \mu_{kl} \Leftrightarrow \omega_i < \omega_k \text{ and } \mu_{ij} < \mu_{kl} \]

Proof. We will prove only the last statement above. Considering the order compactifying family as in Theorem 32, \( \mu_{ij} < \mu_{kl} \Leftrightarrow \mathcal{F}_i \vee \mathcal{F}_j \vee \mathcal{F}_k \vee \mathcal{F}_l \exists \mathcal{F}_m \) \( \mathcal{F}_i \vee \mathcal{F}_j \vee \mathcal{F}_k \vee \mathcal{F}_l \exists \mathcal{F}_m \)

Conversely, if \( \mu_{ij} < \mu_{kl} \) then \( \mathcal{F}_i \vee \mathcal{F}_j \vee \mathcal{F}_k \vee \mathcal{F}_l \exists \mathcal{F}_m \) \( \mathcal{F}_i \vee \mathcal{F}_j \vee \mathcal{F}_k \vee \mathcal{F}_l \exists \mathcal{F}_m \)

The other statements can be proved similarly, using the fact that \( \mathcal{W}^*(x) = \mathcal{W}^*(x) \).

34. Theorem. If \( X^* \) and \( X^{**} \) are finite-point order compactifications of \( X \), and \( X^* \) is the order compactification of \( X \) corresponding to the order compactifying family described in Lemma 33, then \( X = \sup\{X^*, X^{**}\} \).
Proof. First we will show that \( X' \succeq X^* \). Since both \( X^* \) and \( X' \) are order compactifications generated by order compactifying families, the associated embeddings \( x^* \) and \( x' \) of \( X \) into the compactification spaces are simply \( \text{id}_X \). Now recalling that \( X' = X \cup \{ \mu_i \} \) and \( X^* = X \cup \{ \omega_i \} \), define \( f : X' \to X^* \) by \( f(x) = x \) for \( x \in X \cap X' \) and \( f(\mu_i) = \omega_i \). Then \( f \circ x = \mu_i = \text{id}_X \), and a routine exercise verifies that \( f \) is continuous and increasing. Similarly, \( X' \succeq X^{**} \).

Suppose \( (X', x') \) is any other upper bound of \( X^* \) and \( X^{**} \). Since \( X' \succeq X^* \) and \( X' \succeq X^{**} \), there exist continuous increasing maps \( \phi^* : X^* \to X' \) and \( \phi^{**} : X^{**} \to X' \) such that \( \phi^* \circ x^* = x' = \text{id}_X \) and \( \phi^{**} \circ x^* = x' = \text{id}_X \). Define \( \phi : X^* \to X' \) by \( \phi(x) = x' \) for \( x \in X \cap X' \), and

\[
\phi(x) = \mu_i \quad \text{where } \begin{cases} \phi^*(x) = \omega_i, \\ \phi^{**}(x) = r_i \end{cases} \quad \text{for } x \in X \cap X' \cap (X').
\]

Observe that for \( x \in X \), \( \phi \circ x^*(x) = x' = \text{id}_X(x) \), so that \( \phi \circ x^* = x' = \text{id}_X \) as needed. It remains to show that \( \phi \) is continuous and increasing.

\( \phi \) is continuous. If \( N \) is an open neighbourhood of \( x \in X \cap X' \) such that \( N \subseteq X \cap X' \), then \( \phi^{-1}(N) = \phi^*-1(N) \) which is open in \( X^* \) since \( \phi^* \) is a homeomorphism. If \( \mu_i \in X \cap X' \), a basic open neighbourhood \( N \) of \( \mu_i \) is of the form \( (G_i \cap H_i) \cup \{ \mu_i \} \) where \( G_i \) is an open set in \( \mathcal{G}_i \) and \( H_i \) is an open set in \( \mathcal{H}_i \). Thus, \( \phi^{-1}(N) = (\phi^{-1}(G_i) \cap \phi^{-1}(H_i)) \cup \phi^{-1}(\mu_i) \). Since \( \phi^{-1}(G_i) = \phi^*(G_i) \) is an open neighbourhood of each of its points, it remains only to show that \( \phi^{-1}(H_i) \) is a neighbourhood of each point of \( \phi^{-1}(\mu_i) \). Since \( \phi^* \circ x^* = x' = \text{id}_X \), we have \( x' = \phi^* x^* = \phi^* \circ x^* \) so that \( \phi^{-1}(G_i) = \phi^{-1}(G_i) \circ \phi^{-1}(G_i) \). Also, noticing that \( \phi^{-1}(\mu_i) = \mu \) where \( \phi^*(\mu) = \omega_i \), we see that \( \phi^{-1}(\mu_i) = \phi^{-1}(\mu_i) = x' \).

Continuity of \( \phi \) implies that \( \phi^{-1}(G_i \cup \omega_i) = \phi^{-1}(G_i) \cup \phi^{-1}(\mu_i) \) is an open neighbourhood of \( \phi^{-1}(\mu_i) \). Similarly, \( \phi^{-1}(H_i) \cup \phi^{-1}(\mu_i) \) is an open neighbourhood of \( \phi^{-1}(\mu_i) \) in \( X' \).

\( \phi \) is increasing. Suppose \( \alpha \preceq \beta \) in \( X^* \). We consider only the case where \( \alpha, \beta \in X \). Then \( \phi(x) = \mu_i \) and \( \phi(y) = \mu_j \) where \( \phi^*(x) = \omega_i, \phi^*(y) = r_i \), \( \phi^{**}(x) = \omega_j \), and \( \phi^{**}(y) = r_j \). Since \( \phi^* \) and \( \phi^{**} \) are increasing and \( \alpha \preceq \beta \), we have \( \omega_i \preceq \omega_j \) and \( r_j \leq r_i \), so by Lemma 3.3, \( \mu_j \preceq \mu_i \).

A simple induction argument shows that the results of 3.3 and 3.4 generalize to any finite set of finite point order compactifications.

Let \( \text{topsup}(X^*, X^{**}) \) represent the supremum with respect to the ordering of \( T_2 \) topological compactifications of \( X \), while \( \text{sup}(X^*, X^{**}) \) represents the supremum with respect to the ordering of \( T_2 \)-ordered order compactifications of \( X \).

3.5. Theorem. If \( (X^*, **) \) and \( (X^{**}, **) \) are finite-point order compactifications of \( (X, \tau, \theta) \) and \( (X', \tau', \theta') \) is defined on the proof of 3.4 guarantees that \( X' \succeq X' \).

Thus, if \( (X^*, **) \) and \( (X^{**}, **) \) are finite-point topological compactifications of \( (X, \tau, \theta) \) that admit orders \( \theta^* \) and \( \theta^{**} \) that make \( (X^*, \tau, \theta^*) \) and \( (X^{**}, \tau, \theta^{**}) \) \( T_2 \)-ordered compactifications of \( X \), then \( \text{topsup}(X^*, **) \) also admits an order that makes it into a \( T_2 \)-ordered compactification of \( X \).
Finite-point order compactifications

IV

If a topological space has an \( n \)-point compactification, then it has a \( k \)-point compactification for each \( k \in [1, n] \) but \( \mathbb{R} \) with the usual topology and order shows that this statement is false for order compactifications.

Let \( (X^*, \kappa^*) \) and \( (X', \kappa') \) be two finite-point order compactifications of \( X \), with \( X^* = X \cup \{ \mu \}_{\mu \in \kappa} \), and \( X' = X \cup \{ \mu \}_{\mu \in \kappa} \), and with \( X \supseteq X^* \). Then there exists a continuous increasing function \( f : X' \to X^* \) such that \( f \circ \kappa = \kappa^* \). If \( f \) is one-to-one, then \( n = m \). Otherwise, there must be an element \( \omega \in X^* \setminus X \) such that \( f^{-1}(\omega) \) has at least two elements. Observe that \( f^{-1}(\omega) \subseteq X' \setminus \kappa' (X) \). Let \( \mu_1 \) be maximal among \( f^{-1}(\omega) \) and let \( \mu_2 \) be maximal among \( f^{-1}(\omega) \setminus \{ \mu_1 \} \). Identify \( \mu_1 \) and \( \mu_2 \) into a single point \( \mu_0 \) and represent this quotient space by \( Q(X'), \tau' \). Let

\[
\theta' = [0' \cup (X' \setminus \{ \mu_1, \mu_2 \})^2] 
\cup \{(\mu_0, x) : x \in X' \setminus \{ \mu_1, \mu_2 \} \}
\cup \{(x, \mu_0) : x \in X' \setminus \{ \mu_1, \mu_2 \} \}
\cup \{(x, \beta) : x, \beta \in X' \setminus \{ \mu_1, \mu_2 \} \}
\cup \{(\mu_0, \mu_k) \}
\]

A somewhat lengthy but routine argument will give the result stated below. Details may be found in ([8], proposition 46).

4.1 Lemma. \( (Q(X'), \tau', \theta') \) is an \( (m - 1) \)-point order compactification of \( (X, \tau, \theta) \) and \( Q(X') \supseteq X^* \).

4.2 Theorem. If \( (X, \tau, \theta) \) has an \( m \)-point order compactification and an \( n \)-point order compactification, then \( (X, \tau, \theta) \) has a \( k \)-point order compactification for each integer \( k \) between \( m \) and \( n \).

Proof. Suppose \( X^* \) and \( X^{**} \) are \( m \)- and \( n \)-point order compactifications of \( X \), respectively, with \( m < n \). Starting with \( X^* \) and sup \( X^*, X^{**} \), repeatedly apply Lemma 4.1. This result is not valid if \( m \) or \( n = \infty \), since \( \mathbb{R} \) with the usual topology and discrete order has 1-point, 2-point, and infinite-point order compactifications, but no \( k \)-point order compactification for \( 3 \leq k < \infty \).

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