Completely regularly ordered spaces versus $T_2$-ordered spaces which are completely regular

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Abstract

Schwarz and Weck-Schwarz have shown that a $T_2$-ordered space $(X, \tau, \leq)$ whose underlying topological space $(X, \tau)$ is completely regular need not be a completely regularly ordered space (that is, $T_{3.5} + T_2$-ordered $\nRightarrow T_{3.5}$-ordered). We show that a completely regular $T_2$-ordered space need not be completely regularly ordered even under more stringent assumptions such as convexity of the topology. One example involves the construction of a nontrivial topological ordered space on which every continuous increasing function into the real unit interval is constant.

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1. Introduction

If an ordered topological space $(X, \tau, \leq)$ is $T_2$-ordered and $\tau$ is completely regular, then must $(X, \tau, \leq)$ be completely regularly ordered? Schwarz and Weck-Schwarz [14] attribute this question to Brüümmer and provide a negative answer. Their example, however,
fails to have a base of order-convex sets. Such a convexity condition is a modest compatibility requirement between the topology and order of an ordered topological space and is widely assumed. This prompted the revised question of whether a $T_2$-ordered space $(X, \tau, \leq)$ with a so-called convex, completely regular topology $\tau$ is necessarily completely regularly ordered. In Section 2 we present examples showing that the answer is still negative, but each example suggests a strengthening of the hypotheses which might then yield an affirmative answer. In Section 3 we construct a “handle space” having two points which cannot be separated by any continuous increasing real-valued function, and use this space to construct a nontrivial ordered space with no nonconstant continuous increasing functions (into the real unit interval). Variations of these spaces are used to show there seem to be no obvious conditions weaker than “completely regularly ordered” which, together with “completely regular and $T_2$-ordered”, are sufficient to imply “completely regularly ordered”. We show that even regularity conditions are not particularly helpful in this context by imitating some classical constructions (see [7,6]; compare also [10]).

A topological space endowed with a partial order $\leq$ is called an ordered topological space or simply an ordered space. In the following we collect some basic definitions and facts from the theory of ordered topological spaces. For further information we refer the reader to [4,8,11,13].

A mapping $f : X \to Y$ between two ordered spaces is increasing (respectively, decreasing) if $x \leq y$ in $X$ implies $f(x) \leq f(y)$ (respectively, $f(y) \leq f(x)$) in $Y$. If $(X, \tau, \leq)$ is an ordered space and $A \subseteq X$, the increasing hull of $A$ in $X$ is $i_X(A) = \{ y \in X : \exists a \in A$ with $a \leq y \}$. If the context is clear, we may write $i(A)$ for $i_X(A)$, and we will write $i(x)$ for $i(\{x\})$. We say $A$ is an upper set if $A = i(A)$. The decreasing hull $d_X(A)$ of a set $A \subseteq X$ and lower sets are defined dually. A monotone set is a set that is either an upper set or a lower set. The open upper (lower) sets of $(X, \tau, \leq)$ form a topology on $X$ denoted $\tau^\uparrow$ ($\tau^\downarrow$). Observe that $c_{\tau^\downarrow}(A)$ is the smallest closed lower set containing $A$. A set $A \subseteq X$ is convex if $A = i(A) \cap d(A)$, or equivalently, if $x, z \in A$ and $x \leq y \leq z$ imply $y \in A$.

We frequently wish to assume some compatibility conditions between the topology and the order of an ordered topological space. A strongly $T_2$-ordered or $(X, \tau, \leq)$ has a convex topology if $\tau^\uparrow \cup \tau^\downarrow$ is a subbase for $\tau$. $(X, \tau, \leq)$ is $T_2$-ordered if $a \not\leq b$ in $X$ implies there exist disjoint neighborhoods $U$ of $a$ and $V$ of $b$, with $U$ being an upper set and $V$ being a lower set (equivalently, the order relation $\leq$ is closed in the topological product $X \times X$). The neighborhoods $U$ and $V$ need not be open. If $a \not\leq b$ in $X$ implies there exist disjoint open neighborhoods $U$ of $a$ and $V$ of $b$, with $U$ being an upper set and $V$ being a lower set, then we say $(X, \tau, \leq)$ is monotonically separated (see [9]; in [12] these spaces were called strongly $T_2$-ordered). An even stronger form of separation of points is the first condition of the definition of completely regularly ordered spaces. $(X, \tau, \leq)$ is completely regularly ordered if (a) for any $a \not\leq b$ in $X$, there exists a continuous increasing real-valued function $h$ on $X$ with $h(a) > h(b)$, and (b) for any $a \in X$ and any closed set $F \subseteq X \setminus \{a\}$, there exist an increasing function $f : X \to [0, 1]$ and a decreasing function $g : X \to [0, 1]$ with $f(a) = 1 = g(a)$ and $\min\{f(x), g(x)\} = 0$ for $x \in F$. In particular, note that a completely regularly ordered space is monotonically separated and has a convex topology.

A $T_2$-ordered space $(X, \tau, \leq)$ is strongly upper regularly ordered (see [12]) if for any closed upper set $F$ and any $x \in X \setminus F$, there exists an open upper set $U$ and an open lower
set $V$ with $U \cap V = \emptyset$, $F \subseteq U$, and $x \in V$. The dual condition defines strongly lower regularly ordered, and $(X, \tau, \leq)$ is strongly regularly ordered if it is both strongly upper and strongly lower regularly ordered. Note that the latter condition exactly means that the associated bitopological space $(X, \tau^\#, \tau^\#)$ is pairwise regular.

An ordered compactification of an ordered topological space $(X, \tau, \leq)$ is a compact $T_2$-ordered space $(X', \tau', \leq')$ which contains a homeomorphic, order isomorphic copy of $(X, \tau, \leq)$ as a dense subspace. It is well known (see, e.g., [4, Corollary 4.10]) that a $T_2$-ordered space admits an ordered compactification if and only if it is completely regularly ordered. A $T_2$-ordered space $(X, \tau, \leq)$ with convex topology such that the bitopological space $(X, \tau^\#, \tau^\#)$ is pairwise completely regular is called (compare [11,9]) strictly completely regularly ordered. Various conditions have been found that imply this strictly stronger version of complete regular orderedness (see [1,9]).

2. Examples

The following two examples show that in general, a completely regular $T_2$-ordered space with convex topology need not be completely regularly ordered. Thus, these examples illustrate that a $T_2$-ordered space with convex topology which admits Hausdorff compactifications need not admit ordered compactifications.

Example 1. Let $X = (\omega_1 + 1) \times [0] \oplus [(\omega_1 + 1) \times (\omega_0 + 1) \setminus \{(\omega_1, \omega_0)\}] \times [1] \oplus (\omega_0 + 1) \times [2]$; that is, $X$ is the topological sum of copies of $\omega_1 + 1$, $\omega_0 + 1$ and the (deleted) Tychonoff plank, all equipped with their usual topology. As a partial order on $X$ choose $\leq = \Delta_X \cup \{(\alpha, 0), (\alpha, \omega_0, 1) \mid \alpha \in \omega_1\} \cup \{(\omega_1, n, 1), (n, 2) \mid n \in \omega_0\}$. Obviously $\leq$ is closed so that $(X, \leq)$ is $T_2$-ordered. Note that each point has a neighborhood base consisting of open upper sets only or of open lower sets only. Hence the topology of $X$ is convex. Since the unique (Hausdorff) compactification of the Tychonoff plank is the one-point-compactification, we obtain the unique compactification $\beta X$ of $X$ as the one-point-compactification by adding the point $(\omega_1, \omega_0, 1)$; use, e.g., the reflection property to see this.

Suppose that $\leq$ is a closed order on $\beta X$ extending the order $\leq$ on $X$. Then clearly $((\omega_1, 0), (\omega_1, \omega_0, 1))$ and $((\omega_1, \omega_0, 1), (\omega_0, 2))$ belong to $\leq$, and by transitivity, we have the contradiction that $(\omega_1, 0) \leq (\omega_0, 2)$. We conclude that $(X, \leq)$ does not have a $T_2$-ordered compactification. Hence $X$ is not completely regularly ordered, although the topology of $X$ is convex and locally compact.

In fact, the proof just given can be simplified: In the space $X$ constructed above, $(\omega_1, 0)$ and $(\omega_0, 2)$ are incomparable, but the usual proof that the Tychonoff plank is not normal shows that each open upper set containing $(\omega_1, 0)$ intersects each open lower set containing $(\omega_0, 2)$. Thus, $X$ is not monotonically separated and hence not completely regularly ordered.

This example suggests that we should revise our question by strengthening the $T_2$-ordered hypothesis to monotonically separated. The revised question is thus: if $X$ is a completely regular, monotonically separated ordered space with a convex topology, is
X completely regularly ordered? In the next section (see Example 4), we will see that the answer is negative, but first let us consider another counterexample to the original question, which suggests another revision of the hypotheses.

**Example 2.** Let $X = [(\omega_1 + 1) \times (\omega_0 + 1) \times \{0\}] \oplus [(\omega_1 + 1) \times (\omega_0 + 1) \times \{2\}]$ where the factor spaces of the two products are equipped with the usual order topologies and $\oplus$, as above, denotes the topological sum. Hence $X$ is the sum of two Tychonoff planks. We obtain the compact Hausdorff quotient space $Y$ from $X$ by identifying $(\alpha, \omega_0, 0)$ with $(\alpha, \omega_0, 2)$ whenever $\alpha \in \omega_1 + 1$. The corresponding nontrivial equivalence classes will be denoted by $(\alpha, \omega_0, 0/2)$ in the following. Delete the point $(\omega_1, \omega_0, 0/2)$ from $Y$ to obtain the subspace $Z$ of $Y$. Construct the space $F$ from $Z$ by adding a copy of a convergent sequence, that is, suppose that $F = Z \oplus [(\omega_1 + 1) \times \{1\}]$. Define a partial order $\leq$ on $F$ as follows: Set $\leq = \Delta_F \cup \{(\omega_1, n, i), (\omega_1, n, j): n \in \omega_0, i \leq j; i, j \in \{0, 1, 2\}\}$. Obviously $F$ is a completely regular space that is $T_2$-ordered by $\leq$. It is readily seen that $F$ has a convex topology.

Note next that for each completely regularly ordered space $X$ and $x \in X$, the collection of all sets $\text{cl}_x I \cap \text{cl}_x D$ where $x \in I \cap D$, $I$ is an open upper set, and $D$ is an open lower set, forms a neighborhood base at $x$. We want to show that $F$ does not satisfy this condition: Let $x = (\omega_1, \omega_0, 1)$ and consider the open neighborhood $G = F \setminus (\omega_1 \times \{\omega_0\} \times \{0/2\})$ of $x$ in $F$. If $x \in I \cap D$, where $I$ is an open upper set and $D$ is an open lower set, then there exist $\alpha \in \omega_1$ and $n_0 \in \omega_0$ such that $[\alpha, \omega_1] \times [n_0, \omega_0] \times \{0\} \subseteq D$ and $[\alpha, \omega_1] \times [n_0, \omega_0] \times \{2\} \subseteq I$. It clearly follows that $(\alpha + 1, \omega_0, 0/2) \in \text{cl}_x I \cap \text{cl}_x D$. We conclude that $F$ is not completely regularly ordered. (It can also be readily seen that the given order cannot be extended to the unique $T_2$-compactification of the space $Z$.)

This example suggests that, perhaps, the hypothesis of convexity in the original question should be replaced by the following stronger condition of regularity, which is clearly satisfied in every completely regularly ordered space: For each $x \in X$, the sets of the form $\text{cl}_x I \cap \text{cl}_x D$ where $x \in I \cap D$, $I$ is an open upper set and $D$ is an open lower set form a neighborhood base at $x$. Note that, for instance, each strongly regularly ordered space with a convex topology satisfies the latter condition. Again the construction of the following section will show (see Examples 3 and 5) that the answer remains negative even with these strengthened hypotheses.

3. The handle space

For the following construction we need a completely regular Hausdorff space $T$ that contains two disjoint closed copies of $\omega_1$ that cannot be separated by disjoint open sets and in which the complement $K$ of the union of the two copies of $\omega_1$ is dense in $T$. In fact in order to simplify the argument we can (and do) also assume that all points in the complement $K$ are isolated.

As a concrete example the reader might wish to consider the space $\Omega = (\omega_1 + 1) \times \omega_1$ with its usual topology. Note first that all the points $(\alpha, \alpha)$ where $\alpha \in \omega_1$ and $\alpha$ is equal to 0 or a successor ordinal are isolated points of $\Omega$. Observe also that the map $h$ from $\omega_1$ to
the set of its nonzero limit ordinals defined by $h(\alpha) = \omega + \omega \cdot \alpha$ (where we use addition and multiplication of ordinals) is a continuous bijection that is also closed; hence $h$ is a homeomorphism. Therefore the subspace $\{(\alpha, \alpha): \alpha \in \omega_1 \setminus \{0\}\}$ is a limit ordinal of the diagonal and the right edge $\{(\omega_1, \alpha): \alpha \in \omega_1\}$ provide us with two appropriate disjoint closed topological copies of $\omega_1$ that cannot be separated by disjoint open sets. Our starting space $T$ is obtained by declaring points not belonging to these two subsets of $\Omega$ isolated and not changing the neighborhoods of the remaining points. In particular the complement of the union of the two chosen copies of $\omega_1$ is dense in $T$.

As another such space $T$ we could choose van Douwen’s gap space (see [15]). This space is even separable, and the reader can readily check that working with this space would yield separable examples in Examples 3, 4, and 5 below.

For $n \in \mathbb{Z}$, let $E_n$ be the topological sum of two copies $(A_n \cup B_n \cup X_n)$ and $(C_n \cup D_n \cup Y_n)$ of $T$, where $A_n$, $B_n$, $C_n$ and $D_n$ are copies of $\omega_1$; $X_n$ and $Y_n$ are distinct copies of $K$, each as a dense set of isolated points in its respective copy of $T$; and the unions of spaces $T$ involved are disjoint unions.

Put $E = \bigoplus_{n \in \mathbb{Z}} E_n$ and $H = E \cup \{\pm \infty\}$, where the basic neighborhoods of $-\infty$ are of form $\{\pm \infty\} \cup \bigcup_{n \leq k} E_n$ and the basic neighborhoods of $\infty$ are of form $\{\infty\} \cup \bigcup_{n \geq k} E_n$.

The order on $H$ is defined as follows. Recall that $A_n$, $B_{n+1}$, $C_{n+1}$ and $D_n$ are copies of $\omega_1$. For every $n \in \mathbb{Z}$, we put the points of $A_n$ pointwise smaller than the points of $B_{n+1}$; the points of $B_{n+1}$ pointwise smaller than the points of $C_{n+1}$; and the points of $C_{n+1}$ pointwise smaller than the points of $D_n$. Thus, any point from a copy of $\omega_1$ in $H$ is contained in a four-element chain. Each point of $\{\pm \infty\} \cup \bigcup_{n \in \mathbb{Z}} (X_n \cup Y_n)$ is incomparable to every other point of $H$. See Fig. 1.

We will call $H$ the handle space, and will now verify some basic properties of $H$.

$H$ is completely regular: If $x \in E_j$ and $F$ is a closed set not containing $x$, let $g: E_j \to [0, 1]$ be a continuous function with $g(x) = 0$ and $g(F \cap E_j) = 1$. Extending $g$ to $H$ by taking $g(x) = 1$ for any $x \in H \setminus E_j$ gives a continuous function on $H$ separating $x$ and $F$.

If $x = \infty$ and $F$ is a closed set not containing $x$, then $H \setminus F$ contains a neighborhood $N = \{\infty\} \cup \bigcup_{n \geq k} E_n$ of $\infty$. The characteristic function on $N$ is a continuous function separating $x$ and $F$. The case for $x = -\infty$ is analogous.

$H$ is $T_2$-ordered: Suppose $(x_\lambda, y_\lambda)_{\lambda \in A}$ is a net in $H^2$ converging to $(x, y)$ and $x_\lambda \leq y_\lambda$ for all $\lambda \in A$. If $x = y$, then $x \leq y$, so assume $x \neq y$. Now $x_\lambda \leq y_\lambda$ implies $x_\lambda = y_\lambda$.

![Fig. 1. The handle space $H$.](image-url)
or \(x_\lambda\) and \(y_\lambda\) are part of a four-element chain contained in \(E_n \cup E_{n+1}\) for some \(n \in \mathbb{Z}\). Thus, if \(x_\lambda \to x \in [\pm \infty]\), then \(y_\lambda \to y = x\) as well, contradicting \(x \neq y\). Now suppose
\[
(x_\lambda, y_\lambda) \to (x, y) \in E^2.
\]
If \(x \in E_j\), then \(x_\lambda\) is eventually in \(E_j\) and without loss of generality, we will assume \(x_\lambda \in E_j\) for all \(\lambda \in \Lambda\). Because the order was defined by putting four copies of \(\omega_1\) pointwise smaller than each other, if \(x_\lambda \to x \in E_j\) and \(x_\lambda \leq y_\lambda\) for all \(\lambda\), then \(y_\lambda\) must be the corresponding net \(x_\lambda\) lifted to a higher copy of \(\omega_1\), and thus the limit \(y\) of \((y_\lambda)\) must be the lifted copy of \(x\), which is above \(x\), as needed.

The topology of \(H\) is convex. Standard neighborhood subbases of any \(x \in E\) can be chosen to consist of open monotone sets; \(-\infty\) has a neighborhood base of open lower (and also upper) sets of form \([-\infty) \cup \bigcup_{n<k} E_n \cup B_k \cup C_k \cup X_k \cup Y_k\) and \(\infty\) has a neighborhood base of open upper (and also lower) sets of form \((\infty) \cup \bigcup_{n>k} E_n \cup A_k \cup D_k \cup X_k \cup Y_k\).

Observe that if a closed lower set \(M\) in an ordered topological space \((X, \tau, \leq)\) can be separated from a closed upper set \(N\) in \(X\) by a continuous increasing function \(f: X \to [0, 1]\) with \(f(M) = 0\) and \(f(N) = 1\), then there exists a countable dense set \(Q \subseteq [0, 1]\) and a collection of open lower sets \([V_p: p \in Q]\) with the properties that \(p < q \implies \text{cl}_{\tau}(V_p) \subseteq V_q\) and \(M \subseteq V_p \subseteq X \setminus N\) for all \(p, q \in Q\). Indeed, taking \(V_p = f^{-1}[0, p]\) for \(p \in Q \cap [0, 1]\) provides the desired collection. Furthermore, the converse holds as well by the Urysohn construction: If \(Q\) is a countable dense subset of \([0, 1]\) and \([V_p: p \in Q]\) is a collection of open lower sets with the properties that \(p < q \implies \text{cl}_{\tau}(V_p) \subseteq V_q\) and \(M \subseteq V_p \subseteq X \setminus N\) for all \(p, q \in Q\), and then \(f(x) = \inf\{p \in Q: x \in V_p\}\) (where as usual \(\inf \emptyset = 1\)) is a continuous increasing function separating \(M\) and \(N\) as desired.

**Proposition 1.** Let \(H\) be the handle space defined above. For any continuous increasing function \(f: H \to [0, 1]\), we have \(f(\infty) = f(-\infty)\). In particular, \(H\) is not completely regularly ordered.

**Proof.** Suppose to the contrary. Then either there exists a continuous increasing function \(f: H \to [0, 1]\) with \(f(-\infty) = 0\) and \(f(\infty) = 1\) or there exists such a function with \(f(-\infty) = 1\) and \(f(\infty) = 0\). We will only consider the former case; the latter case is analogous. Now \(M = f^{-1}[0, 0.5]\) and \(N = f^{-1}[0.5, 1]\) are lower and upper sets, respectively. Let \(k\) be the largest integer such that \([-\infty) \cup \bigcup_{n<k} E_n \subseteq M\). Similarly, let \(r\) be the smallest positive integer such that \(E_{k+r} \subseteq N\). Since \(M\) is a lower set and \(D_k \subseteq E_k \subseteq M\), we have that \(C_{k+1}\) is a subset of \(M\).

By the well-known result that each bounded real-valued function defined on the topological space \(\omega_1\) is constant on a final segment of \(\omega_1\) [3, Example 3.1.27], \(f\) must be constant on a final segment of \(D_{k+1}\) and on a final segment of \(C_{k+1}\). If these constant values were not equal, for any \(m\) strictly between the values, \(f^{-1}[0, m]\) and \(f^{-1}[m, 1]\) would give open sets separating final segments of \(D_{k+1}\) and \(C_{k+1}\), contrary to the non-normality of the space \(T\). Thus, \(f\) has the same value on final segments of \(D_{k+1}\) and \(C_{k+1} \subseteq M = f^{-1}[0, 0.5]\), so a final segment of \(D_{k+1}\) is contained in \(M\).

Iterating this argument, we see that \(M\) contains a final segment of \(D_{k+r}\). We have reached another contradiction, since \(M\) is disjoint from \(N\) and \(N\) was supposed to contain all of \(E_{k+r}\).
We also note that if the left half of the handle space is deleted, leaving \( H^+ = \bigoplus_{n \in \mathbb{N}} E_n \cup \{\infty\} \), we still cannot separate \( \infty \) from the closed set \( C_1 \) by a continuous increasing real-valued function.

**Example 3.** The handle space \( H \), by Proposition 1, is not completely regularly ordered, but \( H \) is a completely regular, \( T_2 \)-ordered space in which each point \( x \) has a neighborhood base of sets of the form \( \text{cl}_y I \cap \text{cl}_z D \) where \( x \in I \cap D \). \( I \) is an open upper set and \( D \) is an open lower set. Since we already know that the topology of \( H \) is convex, the last statement is a consequence of the following proof which shows that \( H \) is strongly regularly ordered:

Indeed, the \( \tau^\flat \)-neighborhoods at any point \( x \in H \) have a neighborhood base consisting of closed lower sets, and dually: If \( x \in X_n \cup Y_n \), then \( \{x\} \) is such a closed lower neighborhood of \( x \). If \( x \) is an element of a copy \( A_n, B_n, C_n \) or \( D_n \) of \( \omega_1 \) and \( d(x) \) has \( n \in \{1, 2, 3, 4\} \) elements, there is a clopen lower neighborhood of \( x \) consisting of copies of a closed interval \([\alpha + 1, x]\) in the \( n \) copies of \( \omega_1 \) below \( x \), together with points from some \( X_j \)'s and some \( Y_j \)'s. If \( x = \infty \), each open lower neighborhood of \( \infty \) of form \( \{\infty\} \cup A_n \cup X_n \cup \bigcup_{k>n} E_k \) contains a closed lower neighborhood of \( \infty \) of form \( \{\infty\} \cup A_n \cup \bigcup_{k>n} E_k \). If \( x = -\infty \), each open lower neighborhood of \( -\infty \) of form \( \{\infty\} \cup \bigcup_{k<n} E_k \cup A_n \cup X_n \cup B_n \cup C_n \cup Y_n \) contains a closed lower neighborhood of \( -\infty \) of form \( \{\infty\} \cup \bigcup_{k<n} E_k \cup B_n \cup C_n \). These neighborhoods show that the \( \tau^\flat \)-neighborhoods of \( x \in H \) have a base of closed lower sets. Similarly one shows that each point in \( H \) has a \( \tau^\flat \)-neighborhood base consisting of closed upper sets.

**Example 4.** It follows from above that the subspace \( X = H \setminus \{-\infty\} \) of the handle space \( H \) is completely regular, \( T_2 \)-ordered, and has a convex topology. We now show that for any two points \( x, y \) in \( X \) such that \( x \neq y \) there exists a continuous increasing function \( f : X \to [0, 1] \) such that \( f(x) > f(y) \). In particular, \( X \) is monotonically separated. Indeed if \( x \neq -\infty \) we can clearly find a clopen upper set \( S \) containing \( x \), but not \( y \). Similarly, if \( y \neq -\infty \) we can choose a clopen lower set \( X \setminus S \) containing \( y \), but not \( x \). Hence in either case it is possible to define the required function as the characteristic function of \( S \).

It may be worthwhile to mention here that \( H \) is also monotonically separated, because any strongly regularly \( T_2 \)-ordered space \( X \) is monotonically separated: In fact, if \( x, y \in X \) and \( x \neq y \), then \( y \neq i(x) \). Since \( i(x) \) is closed and \( X \) is strongly upper regularly ordered, there are open sets witnessing monotonic separation of \( X \).

A completely regularly ordered space has “enough” continuous monotone (that is, increasing or decreasing) real-valued functions to separate points and to separate points from closed sets, as described in the definition of “completely regularly ordered”. The motivating question for this paper asks whether a \( T_2 \)-ordered space with a convex topology and enough continuous functions to separate points from closed sets already has “enough” continuous monotone real-valued functions. Our final example shows emphatically that these conditions are inadequate; such an ordered space may have no continuous increasing functions into the unit interval \([0, 1]\) other than the constant functions, even if the associated bitopological space is pairwise regular.

**Example 5.** We shall use the notation of the discussion of the handle space \( H \).
Let $S_0 = H$ and $D = \bigcup_{n \in \mathbb{Z}} (X_n \cup Y_n)$. Observe that $D$ is a dense subset of isolated points in $H$, $\pm\infty \notin D$, and each pair of elements of $D$ is incomparable. Let

$$S_1 = S_0 \oplus \bigoplus_{(a,b) \in D^2, a \neq b} H(a,b)$$

where for any $(a,b) \in D^2$ with $a \neq b$, $H(a,b) = H(a,b,1) \oplus H(a,b,2)$ where $H(a,b,i) = H$ for $i \in \{1, 2\}$. We will call each $H(a,b)$ a double handle. As a labeling convention, $x \in H(a,b,i)$ will be denoted $x_{(a,b,i)}$. Note that $S_1$ is a completely regular Hausdorff space as the topological sum of completely regular Hausdorff spaces.

Order $S_1$ by keeping the order on $S_0$ and on each $H(a,b,i)$, $a$ and $b \in D$, $a \neq b$, putting

$$-\infty_{(a,b,1)} \geq a \geq -\infty_{(a,b,2)}$$

and

$$\infty_{(a,b,1)} \leq b \leq \infty_{(a,b,2)}$$

and any additional order between points $\pm \infty_{(a,b,i)}$ required by transitivity. Observe that on $S_0$ and on the added handles the new partial order agrees with the original one. See Fig. 2.

$S_1$ is $T_2$-ordered: We show that the graph of the order on $S_1$ is closed in the product $S_1 \times S_1$. If $x_{\lambda} \leq y_{\lambda}$ for all $\lambda \in \Lambda$ and $x_{\lambda} \to x$, $y_{\lambda} \to y$, then since $S_1$ consists of a large topological sum, eventually $x_{\lambda}$ is in one summand and eventually $y_{\lambda}$ is in one summand. If these nets are eventually in the same summand, then $x \leq y$ since the order on each summand is closed. If $x_{\lambda}$ and $y_{\lambda}$ are eventually in distinct summands $H(a,b,i)$ and $H(c,d,j)$ or in $S_0$, then it follows that the net $(x_{\lambda}, y_{\lambda})$ is constantly one of the pairs $(-\infty_{(a,b,2)}, a)$, $(a, -\infty_{(a,b,1)})$, $(\infty_{(a,b,1)}, b)$, or $(b, \infty_{(a,b,2)})$ used to define the order on $S_1$, or (by transitivity), a pair of form $(-\infty_{(a,b,2)}, \infty_{(c,a,2)})$, $(-\infty_{(a,b,2)}, -\infty_{(a,c,1)})$, $(\infty_{(b,a,1)}, \infty_{(c,a,2)})$, or $(\infty_{(b,a,1)}, -\infty_{(a,c,1)})$ for appropriate points $a, b, c \in D$. As an eventually constant net in the graph of the order, the limit is also in the graph of the order.

Turning to the convexity of the topology on $S_1$, let $N$ be an open neighborhood of $-\infty_{(a,b,2)}$ contained in $H(a,b,2) \setminus \{\infty_{(a,b,2)}\}$. Without loss of generality, $N$ is an open lower neighborhood of $-\infty_{(a,b,2)}$ in the handle $H(a,b,2)$ (see the discussion above showing that $H$ has a convex topology), and for such a neighborhood $N$, we have $d(N) = N$ in $S_1$, and it follows that $\infty_{(a,b,2)}$ has a subbase of open monotone sets in $S_1$. Similar arguments show that each point of $H(a,b,1) \cup H(a,b,2)$ has a neighborhood subbase of open monotone sets.

![Fig. 2. Adjoining double handles.](image-url)
For each subset \( U \) of \( S_0 \) we have
\[
i(U) = i_{S_0}(U) \cup \{ \infty(a, b, 2) \in S_1 : b \in U \} \cup \{ -\infty(a, b, 1) \in S_1 : a \in U \},
\]
\[
d(U) = d_{S_0}(U) \cup \{ -\infty(a, b, 2) \in S_1 : a \in U \} \cup \{ \infty(a, b, 1) \in S_1 : b \in U \}.
\]

Since \( H \) is monotonically separated and \(-\infty\) and \( \infty \) are not comparable in \( H \), there are open lower neighborhoods \( L_\infty \) and \( L_{-\infty} \) of \( \infty \) and \(-\infty\), respectively, and open upper neighborhoods \( I_\infty \) and \( I_{-\infty} \) of \( \infty \) and \(-\infty\), respectively, such that \( L_\infty \cap I_{-\infty} = \emptyset \) and \( L_{-\infty} \cap I_\infty = \emptyset \).

Consider now \( L \cap I \) where \( L \) and \( I \) are an open lower set and an open upper set of \( S_0 \), respectively. According to the formulas above, \( d(L) \) and \( i(I) \) will not be open in general, but
\[
L' = L \cup \bigcup \{ L_{-\infty(a,b,2)} : -\infty(a,b,2) \in S_1, a \in L \}
\]
\[
\cup \bigcup \{ L_{\infty(a,b,1)} : \infty(a,b,1) \in S_1, b \in L \}
\]
and
\[
I' = I \cup \bigcup \{ I_{\infty(a,b,2)} : \infty(a,b,2) \in S_1, b \in I \}
\]
\[
\cup \bigcup \{ I_{-\infty(a,b,1)} : -\infty(a,b,1) \in S_1, a \in I \}
\]
will be an open lower set and an open upper set in \( S_1 \), respectively. (Here we are using a self-explanatory notation to denote copies of neighborhoods in the added handles.)

Clearly, \( L' \cap I' = L \cap I \). Hence, we finally conclude that each point of \( S_0 \) has a subbase of open monotone sets in \( S_1 \), because the topology of the subspace \( S_0 \) had the corresponding property. We have shown that \( S_1 \) has a convex topology.

Now observe that any continuous increasing function \( f \) on \( S_1 \) into the unit interval \([0, 1]\) must be constant on the dense subset \( D \) of \( S_0 \). If not, then there exist \( a, b \in D \), with \( f(a) \neq f(b) \). However, we have (referring to Fig. 2)
\[
f(a) \geq f(-\infty(a,b,2)) = f(\infty(a,b,2)) \geq f(b)
\]
\[
\geq f(\infty(a,b,1)) = f(-\infty(a,b,1)) \geq f(a)
\]
so that \( f(a) = f(b) \), a contradiction.

As \( f \) is constant on \( D \) and \( D \) is dense in \( S_0 \), we find that any continuous increasing real-valued function on \( S_1 \) is constant on \( S_0 \).

Without further modifications, such functions may be nonconstant on \( H(a,b,i) \setminus \{ \pm \infty(a,b,i) \} \). To rectify this, we iterate the process, forming a space \( S_{n+1} \) from \( S_n \) by adding double handles between each pair of points in the copy of the dense subset \( D \) of each handle added in the construction of \( S_n \). Then each continuous real-valued function defined on \( S_{n+1} \) will be constant on \( S_n \). The proof is by induction on \( n \). Indeed any continuous increasing real-valued function on \( S_{n+1} \) must be constant, say equal to \( r \), on \( S_{n-1} \) by our induction hypothesis. Furthermore on each handle \( H \) belonging to \( S_n \setminus S_{n-1} \) such a function must be equal to some constant \( r_H \) because of the new handles added in \( S_{n+1} \setminus S_n \) and the argument given above. But the values of such a function on the points \( +\infty \) and \(-\infty \) of \( H \) must be equal to the values on the two points of \( S_{n-1} \) from which that handle \( H \) originated so that
We conclude that any increasing real-valued function on $S_{n+1}$ must be constant on $S_n$.

After countably many iterations, we get a space $S_\infty$ with the property that every continuous increasing real-valued function into the unit interval $[0, 1]$ is constant. Specifically, let $S_\infty = \bigcup_{n=0}^{\infty} S_n$, and take the union of the topologies on each $S_n$ as a base for the topology on $S_\infty$. Similarly we define the order on $S_\infty$ as the union of the orders on the subsets $S_n$.

The space $S_\infty$ is a topological sum $\bigoplus_{\lambda \in \Lambda} H_\lambda$ of copies $H_\lambda$ ($\lambda \in \Lambda$) of the handle space $H$. Let $\infty_1$ and $-\infty_1$ denote the endpoints of the handle $H_\lambda$, let $R_\lambda$ denote the copy of $R = \bigcup_{\mu \in \mathbb{Z}} (A_\mu \cup B_\mu \cup C_\mu \cup D_\mu)$ in $H_\lambda$, and let $L_\lambda$ denote the copy of $D = \bigcup_{\mu \in \mathbb{Z}} (X_\mu \cup Y_\mu)$ in $H_\lambda$. The order on $S_\infty$ consists of 4-element chains involving points of $R_\lambda$ within any given handle $H_\lambda$ in $S_\infty$, together with 3-element chains of form

$$-\infty(a,b,1) \geq a \geq -\infty(a,b,2) \quad \text{or} \quad \infty(a,b,1) \leq b \leq \infty(a,b,2)$$

linking points $a, b \in D_\lambda \subseteq S_n$ to the endpoints of handles in the next higher iteration. Note that each element of $R_\lambda$ is contained in a single 4-element chain, and each element $\pm \infty(a,b,i) \in H_\lambda \setminus S_0$ is contained in a single 3-element chain, but elements $a, b \in D_\lambda$ are the middle points of infinitely many 3-element chains.

From the arguments above, each $S_n$ and $S_\infty$ is $T_2$-ordered.

As a topological sum of completely regular spaces, $S_\infty$ is completely regular. However, $S_\infty$ is clearly not completely regularly ordered, for no pair of distinct points can be separated by a continuous increasing real-valued function.

By induction it follows from the convexity argument given above, that each point of any $S_n$ has a subbase of monotone open sets in $S_n$. Furthermore any open lower set $L_n$ and any open upper set $I_n$ of $S_n$ can be (canonically, see above) extended to an open lower set $L_{n+1}$ and an open upper set $I_{n+1}$ of $S_{n+1}$, respectively, such that $L_{n+1} \cap I_{n+1} = L_n \cap I_n$. Then $L_\infty = \bigcup_{k \geq n} L_k$ and $I_\infty = \bigcup_{k \geq n} I_k$ are an open lower set and an open upper set in $S_\infty$, respectively, such that $L_\infty \cap I_\infty = L_\infty \cap I_\infty$. Indeed if $x \in L_L \cap I_I$ then $x \in L_m \cap I_m$ where $m = \max(s, t)$. But $L_m \cap I_m = L_m \cap I_m$ by induction. We conclude that each point of $S_\infty$ has a subbase of open monotone sets in $S_\infty$. Therefore we have shown that $S_\infty$ has a convex topology.

Now we will show that $S_\infty$ is strongly regularly ordered. Suppose $U$ is an open upper neighborhood of $x \in H \subseteq S_{n} \setminus S_{n+1}$, where we use the convention that $S_{-1} = \emptyset$. Let $M_n$ be an open (in $H_\lambda$) upper (in $H_\lambda$) neighborhood of $x$, and let $N_m$ be a closed (in $H_\lambda$) upper (in $H_\lambda$) neighborhood of $x$, with $M_n \subseteq N_m \subseteq H_\lambda \cap U$. The existence of such neighborhoods follows from the remarks of Example 3. Now $i(M_n) = M_n \cup P M_n$ where

$$PM_n = \{-\infty(a,b,1) : a \in M_n \cap D_\lambda, b \in D_\lambda \text{ for some } \lambda_0\} \cup \{\infty(a,b,2) : b \in M_n \cap D_\lambda, a \in D_\lambda \text{ for some } \lambda_0\},$$

and $i(N_n) = N_n \cup P N_n$ where

$$PN_n = \{-\infty(a,b,1) : a \in N_n \cap D_\lambda, b \in D_\lambda \text{ for some } \lambda_0\} \cup \{\infty(a,b,2) : b \in N_n \cap D_\lambda, a \in D_\lambda \text{ for some } \lambda_0\}.$$

Now $i(M_n)$ may not be open. We will carefully add segments around each point of $PM_n$ to make it open, but the resulting set may not be an upper set, so we iterate this
process on $M_n$ and $N_n$. It is important to note that each $z \in P M_n$ (or $P N_n$) is on a separate handle in $S_{n+1} \setminus S_n$. For $z \in P M_n$, let $H_{z_i}$ be the handle containing $z$ and pick an open (in $H_{z_i}$) upper (in $H_{z_i}$) neighborhood $M_z$ of $z$, contained in a closed (in $H_{z_i}$) upper (in $H_{z_i}$) neighborhood $N_z$ of $z$, with $M_z \subseteq N_z \subseteq H_{z_i} \cap U$. Now put

$$M_{n+1} = \bigcup_{z \in P M_n} M_z \text{ and } N_{n+1} = \bigcup_{z \in P N_n} N_z.$$ 

Note that $M_{n+1} \subseteq N_{n+1} \subset S_{n+1} \setminus S_n \subset U$. $M_{n+1}$ is a union of open sets ($\theta$ or $M_z$) from each summand of the topological sum $S_\infty$, and thus is open in $S_\infty$. Similarly, $N_{n+1}$ is closed. Furthermore, since the summands used in $N_n$ are distinct from those used in $N_{n+1}$, $N_n \cup N_{n+1}$ is closed.

We may now iterate the process: Take $i(M_n \cup M_{n+1})$. The “new” points in $i(M_n \cup M_{n+1}) \setminus (M_n \cup M_{n+1})$ are points of form $\pm \infty$ from handles in $S_{n+2} \setminus S_{n+1}$. Choose an open upper (in its handle $H_z$) neighborhood $M_z$ of each such new point $z$, with $M_z$ contained in a closed upper (in its handle) neighborhood $N_z$ of $z$, with $N_z \subseteq H_z \cap U$. Let $M_{n+2}$ be the union of all such $M_z$‘s and $N_{n+2}$ be the union of all such $N_z$‘s added at this iteration. Again, $M_n \cup M_{n+1} \cup M_{n+2}$ consists of an open segment from each of the summands of $S_\infty$ and is thus open. Similarly, $N_n \cup N_{n+1} \cup N_{n+2}$ is closed. Continuing and noting that if $V \subseteq H_z$ is an upper set in $H_z$ where $H_z$ was a handle added in the $n$th iteration $S_n \setminus S_{n-1}$, then $i(V)$ only contains points of $S_{n+1}$, and that the added points are from distinct handles, we see that the process avoids “clustering” of the infinite union of closed sets in $N_k (k \geq n)$ to a set which is not closed. Thus $M = \bigcup_{k \geq n} M_k$ and $N = \bigcup_{k \geq n} N_k$ provide an open upper neighborhood $M$ of $x$ contained in a closed upper neighborhood $N$ of $x$ contained in $U$. Now $M$ and $S_\infty \setminus N$ provide monotone open sets separating $x$ and $S_\infty \setminus U$, showing that $S_\infty$ is strongly lower regularly ordered. The dual argument shows that $S_\infty$ is strongly upper regularly ordered.

**Remark 1.** After this article had been completed, the authors discovered that already several years before Schwarz and Weck-Schwarz, Hommel [5, Remarks 1.3.2(2) and 2.2.3(1)] discussed two examples of $T_2$-ordered spaces with completely regular topology that are not completely regularly ordered. Another such space due to Saint-Raymond was later studied in an article of Edwards [2, p. 74]. However none of these examples has a convex topology.

Hommel’s article also contains a remarkably simple idea to construct a completely regularly ordered space that is not strictly completely regularly ordered (compare [9,10]). Since some inaccuracies in [5] obscure the argument, we close with the description of such an example.

**Example 6 (see Hommel [5, 2.2.3(3)]).** Let $X = [0, 1] \times [0, 1]$ be equipped with its usual compact topology and ordered by $(x_1, y_1) \leq (x_2, y_2)$ provided that $x_1 \leq x_2$ and $y_1 = y_2$. Then the partial order is closed, and since compact $T_2$-ordered spaces are completely regularly ordered [13], we have defined a completely regularly ordered space. Consider the subspace $E = ([0, 1] \times ([0, 1] \cap \mathbb{Q})) \cup F$ where $F = \{1\} \times ([0, 1] \setminus \mathbb{Q})$ of $X$. Of course, $E$ is a completely regularly ordered space. Furthermore $F$ is a closed lower set in $E$. Evidently, there does not exist an open upper set containing $(0, 0)$ and an open lower set containing
$F$ that are disjoint in $E$. Hence $E$ is not strongly regularly ordered. It is also interesting to note that $E$ is an $I$-space (that is, $d(G)$ and $i(G)$ are open whenever $G$ is open in $E$).

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References