On The Operator Equation $AX - XB = C$ With Unbounded Operators A, B and $C^{\ddagger}$.

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Abstract

We find the criteria for the solvability of the operator equation

$$AX - XB = C, \tag{0.1}$$

where A, B and C are unbounded operators, and use the result to show existence and regularity of solutions of nonhomogenuous Cauchy problems.

1 Introduction

Let A and B be operators on Banach spaces E and F, respectively, and C be an operator from F to E. Of concern is the operator equation

$$AX - XB = C. \tag{1.1}$$

To be found is a bounded operator X from F to E such that $X(D(B) \cap D(C)) \subseteq D(A)$ and $AXf - XBf = Cf$ for every $f \in D(B) \cap D(C)$. Over the last few decades, Equation (1.1) has been considered by many authors. It was first studied intensively for bounded operators by Daleckii and Krein [2], Rosenblum [12] (see also [5]). For unbounded operators A and B, the case when A and B are generators of C_0-semigroups was considered in [1], [4] and [13]. Recently, many papers apply the results to the stability and regularity of solutions of the abstract differential equation

$$u'(t) = Au(t) + f(t),$$

(see [13], [15] and [16]), and the higher differential equation

$$u^{(n)}(t) = Au(t) + f(t)$$

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(see [8], [17]). On the other hand, it seems that there is little consideration of Equation (1.1) when C is an unbounded operator.

In this paper, we study Equation (1.1) for this case. The motivation behind this is that, if X is a bounded solution of (1.1), then the operator $\mathcal{A}' := \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is similar to the operator $\mathcal{A} := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ by the identity

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & -X \\ 0 & I \end{pmatrix}.$$

Therefore, many properties of $\mathcal{A}$ can be carried over to $\mathcal{A}'$. Thus, instead of studying the unbounded perturbed operator $\mathcal{A}'$ we just study operator $\mathcal{A}$, which seems to be much simpler. In particular, the operator $\mathcal{A}'$ is a generator of a C_0-semigroup on $E \times F$ if and only if $\mathcal{A}$ is. It is very useful, when converting the nonhomogenous Cauchy problem

$$\begin{cases} u'(t) = Au(t) + f(t), \ t \geq 0, \\ u(0) = x_0 \in E, \end{cases}$$

where f is a vector of a function space $F(\mathbb{R}, E)$, into a homogenous problem

$$\begin{cases} \mathcal{U}'(t) = \begin{pmatrix} A & \delta_0 \\ 0 & \frac{d}{dt} \end{pmatrix} \mathcal{U}(t), \ t \geq 0, \\ \mathcal{U}(0) = (x_0, f) \end{cases}$$

on $E \times F(\mathbb{R}, E)$ (see [6], [7] and [9]). Note that the operator δ_0 is unbounded in some function space $F(\mathbb{R}, E)$.

We organize this paper as follows: In Section 2, we first show the solvability of Equation (1.1). Then we give some applications to the existence and regularity of solutions of the nonhomogeneous Cauchy problem. In Section 3, we consider the nonhomogeneous differential equation

$$u'(t) = Au(t) + f(t) \tag{1.2}$$

on the line $\mathbb{R}$, where $f \in L^p(\mathbb{R}, E)$. It turns out that the existence and unqueness of the bounded mild solution of (1.2) is equivalent to the solvability of equation $AX - X\mathcal{D} = \delta_0$. (See the notations below).

Let us fix some notations. Let E be a Banach space. The value of a functional $\phi \in E^*$ at a vector $x \in E$ is denoted by $< x, \phi >$. By $W^{p,1}(\mathbb{R}, E)$ we denote the space of all absolutely continuous functions f from $\mathbb{R}$ to E with $f' \in L^p(\mathbb{R}, E)$. If $F(\mathbb{R}, E)$ is a certain function space over E, then $\mathcal{D} : D(\mathcal{D}) \subseteq F(\mathbb{R}, E) \to F(\mathbb{R}, E)$ is defined by $\mathcal{D}f = f'$ and $\delta_0 : D(\delta_0) \subseteq F(\mathbb{R}, E) \to E$ by $\delta_0(f) = f(0)$. Finally, for $\lambda \in \varrho(A)$, $(\lambda - A)^{-1}$ is denoted by $R(\lambda, A)$.

2 Solution of the equation $AX - XB = C$.

Throughout this paper, A and $(-B)$ will denote generators of C_0-semigroups $(T(t))$ and $(S(t))$ on Banach spaces E and F, respectively, and C is an operator from F to E. For the operator equation

$$AX - XB = C, \tag{2.1}$$

let us recall a definition. Let B be a linear operator on F. Then we say that $C : F \to E$ is B-*bounded* if $D(B) \subseteq D(C)$ and the operator $C(\lambda - B)^{-1}$ is bounded for one (all) $\lambda \in \varrho(B)$. For example, if C is a closed operator with $D(B) \subseteq D(C)$, then C is B-bounded. We have the following theorem about the solvability of Equation (2.1).

Theorem 2.1 *Let A and $-B$ be generators of C_0-semigroups $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ on E and F, respectively, and C be B-bounded. Let*

$$Q(t) : F \supseteq D(B) \mapsto E : Q(t)f := T(t)CS(t)f,\ t \geq 0, \tag{2.2}$$

and

$$R(t) : F \supseteq D(B) \mapsto E : R(t)f := -\int_0^t Q(s)f ds,\ t \geq 0. \tag{2.3}$$

We assume that

(i) the weak-topology closure of $\{Q(t)f\}_{t\geq 0}$ contains 0 for every $f \in D(B)$;

(ii) $R(t)$ can be extended to a bounded operator for every $t \geq 0$ and family $\{R(t)\}_{t\geq 0}$ is relatively compact in the weak operator topology.

Then Equation (2.1) *has a bounded solution.*

Conversely, if (2.1) *has a bounded solution, then* $R(t)$ *is bounded for every* $t \geq 0$. *In addition, if for every bounded operator* Y *from* F *to* E, $T(t)YS(t)$ *converges to* 0 *as* $t \to \infty$ *in the weak (strong, uniform) operator topology, then the solution* X *of* (2.1) *is unique, and* $R(t)$ *converges to* X *in the weak (strong, uniform) operator topology.*

Remark The operator $R(t)$ is meaningful for each $t \geq 0$, since the function $t \to Q(t)f = T(t)CS(t)f = T(t) \cdot CR(\lambda, B) \cdot S(t) \cdot (\lambda - B)f$ is continuous for each $f \in D(B)$ and $\lambda \in \varrho(B)$.

Proof. Let $\lambda \in \varrho(B)$ and take $C_1 = CR(\lambda, B)$. Then C_1 is bounded. For $t \geq 0$, we define operators $Q_1(t) : F \to E$ and $R_1(t) : F \to E$ by

$$Q_1(t)f := T(t)C_1S(t)f$$

and

$$R_1(t)f := -\int_0^t Q_1(s)f ds.$$

Then $Q_1(t)$ and $R_1(t)$ are bounded operators. We now consider the operator equation

$$AY - YB = C_1. \tag{2.4}$$

By assumptions, there exists a net $t_\alpha \to \infty$ such that $T(t_\alpha)CS(t_\alpha)$ converges weakly to 0 and $R(t_\alpha)$ converge weakly to a bounded operator Q. Therefore, $T(t_\alpha)C_1S(t_\alpha)$ converges weakly to 0 and $R_1(t_\alpha)$ converge weakly to the bounded operator $QR(\lambda, B)$. By [13, Theorem 3], Equation (2.4) has a bounded solution, namely $Y = QR(\lambda, B)$. It implies that $Y(\lambda - B)$ can be extended to the bounded operator Q. We verify that $Q = \overline{Y(\lambda - B)}$ is a solution of (2.1).

First, for any $f \in D(B^2)$ we have $(\lambda - B)f \in D(B)$ and

$$\begin{aligned} AQf - QBf &= AY(\lambda - B)f - Y(\lambda - B)Bf \\ &= (AY - YB)(\lambda - B)f \\ &= C_1(\lambda - B)f = Cf. \end{aligned}$$

Hence, $AQf - QBf = Cf$ and thus,

$$T(t)AQS(t)f - T(t)QS(t)Bf = T(t)CS(t)f$$

for all $f \in D(B^2)$. By [13, Lemma 1], if $f \in D(B^2)$ and $\phi \in D(A')$ we have

$$\begin{aligned} \frac{d}{dt} < T(t)QS(t)f, \phi > &= < T(t)AQS(t)f, \phi) - (T(t)QS(t)Bf, \phi > \\ &= < T(t)CS(t)f, \phi > . \end{aligned}$$

Therefore,

$$\begin{aligned} < R(t)f, \phi > &= -\int_0^t < T(s)CS(s)f, \phi > ds \\ &= -\int_0^t \frac{d}{ds} < T(s)QS(s)f, \phi > ds \\ &= - < T(t)XS(t)f - Qf, \phi >, \end{aligned}$$

from which it follows

$$R(t)f = Qf - T(t)QS(t)f \tag{2.5}$$

for $f \in D(B^2)$. Since the operators on both sides of (2.5) are bounded and $D(B^2)$ is dense in F, it implies (2.5) also holds for all $f \in D(B)$.

Let now $f \in D(B)$ and $\phi \in D(A')$, then we have

$$\begin{aligned} < T(t)CS(t)f, \phi > &= \frac{d}{dt}\int_0^t < T(s)CS(s)f, \phi > ds \\ &= -\frac{d}{dt} < R(t)f, \phi > \\ &= \frac{d}{dt} < T(t)QS(t)f - Qf, \phi > \\ &= \frac{d}{dt} < T(t)QS(t)f, \phi > \\ &= < T(s)AQS(s)f - T(s)QS(s)Bf, \phi >, \end{aligned}$$

which implies

$$T(t)AQS(t)f - T(t)QS(t)Bf = T(t)CS(t)f \text{ for all } t \geq 0.$$

Taking $t = 0$ we have $AQf - QBf = Cf$ for $f \in D(B)$.

Conversely, if X is a solutiond of (2.1), then by the same argument as above we have

$$R(t)f = Xf - T(t)XS(t)f \tag{2.6}$$

for $f \in D(B)$. Since all the operators on the right hand side of (2.6) are bounded and $D(B)$ is dense in F, $R(t)$ can be extended to a bounded operator. Moreover, if $T(t)XS(t) \to 0$ in weak (resp. strong, uniform) operator topology, then $R(t) \to X$ weakly (resp. strongly, uniformly). Hence, X is uniquely determined, and the proof is complete. ♣

For a semigroup $(T(t))_{t\geq 0}$ generated by A, we define *the growth bound* $\omega(A)$ by

$$\omega(A) := inf\{\lambda \in \mathbb{R} : \exists M > 0 \ s.t. \ \|T(t)\| \le Me^{\lambda t} \ \forall t \geq 0\}.$$

If $\omega(A) < 0$, then $(T(t))$ is called *uniformly exponentially stable*. The following is a short version of Theorem 2.1, which gives the existence and uniqueness of the solution of Equation (2.1) and will be used more frequently.

Theorem 2.2 *Assume that* $\omega(A) + \omega(-B) < 0$ *and that* $R(t)$ *is uniformly bounded. Then Equation* (2.1) *has a unique bounded solution.*

Proof. Since $AX - XB = (A+\lambda)X - X(B+\lambda)$, we can assume, without loss of generality, that $\omega(S) = 0$ and $\omega(T) < 0$. Then for any $\lambda \in \varrho(B)$ we have

$$\begin{aligned} \|T(t)CS(t)f\| &= \|T(t)CR(\lambda, B)S(t)(\lambda - B)f\| \\ &\le M_1 e^{\omega(A)t} \cdot \|CR(\lambda, B)\| \cdot M_2 \cdot \|(\lambda - B)f\| \to 0. \end{aligned}$$

So (i) in Theorem 2.1 is satisfied. In addition, for $t_1, t_2 \to \infty$ and $f \in D(B)$ we have

$$\begin{aligned} \|R(t_1)f - R(t_2)f\| &\le \int_{t_1}^{t_2} \|T(s)CS(s)f\| ds \\ &\le \int_{t_1}^{t_2} M_1 e^{\omega(A)s} \|CR(\lambda, B)\| M_2 \|(\lambda - B)f\| ds \\ &\le M \int_{t_1}^{t_2} e^{\omega(A)s} ds \to 0 \text{ as } t_1, t_2 \to \infty. \end{aligned}$$

Since $R(t)$ is uniformly bounded and $D(B)$ is dense in F, $R(t)$ converges strongly to a bounded operator. So (ii) in Theorem 2.1 is satisfied. By Theorem 2.1, (1.1) has a solution, and since $\|T(t)YS(t)f\| \to 0$, $t \to \infty$ for each bounded operator $Y : F \to E$, it is unique and equals to the bounded extension of $\lim_{t\to\infty} R(t)$. ♣

The following corollary, which is involved with *exponentially dichotomic* semigroups follows directly from Theorem 2.2. Recall, a C_0-semigroup $(T(t))$ on Banach space E is exponentially dichotomic, if there is a bounded projection P on E and positive constants, M and ω, such that

(i) $PT(t) = T(t)P$ for all $t \geq 0$;

(ii) $\|T(t)x\| \square Me^{-\omega t}\|x\|$ for all $x \in P(E)$;

(iii) The restriction $T(t)|ker(P)$ extends to a group and $\|T(-t)x\| \square Me^{-\omega t}\|x\|$ for all $x \in ker(P)$ and $t \geq 0$.

It is well-known that $(T(t))$ is exponentially dichotomic if and only if $\sigma(T(t)) \cap \{\lambda \in \mathbb{C} : |\lambda| = 1\} = \emptyset$ for all $t > 0$ (see e.g. [10]). By the weak spectral mapping theorem, this implies $\sigma(A) \cap i\mathbb{R} = \emptyset$. Also note that uniformly exponentially stable semigroup is a particular case of exponentially dichotomic one, when $P = I$, the identity operator.

Corollary 2.3 *Let A be the generator of an exponentially dichotomic semigroup $(T(t))$ and $-B$ be the generator of an isometric C_0-group. If $R(t)$ is uniformly bounded. Then Equation* (2.1) *has a unique bounded solution.*

Proof. Let $C_1 = PC$, $C_2 = (I - P)C$, $A_1 = A|P(E)$, $A_2 = A|ker(P)$. It is easy to see that C_1 and C_2 are $B -$ *bounded*. By Theorem 2.2, there is a unique bounded operator $X_1 : F \to P(E)$ such that

$$A_1X_1 - X_1B = C_1. \tag{2.7}$$

Moreover, $-A_2$ generates an exponentially stable semigroup and B also generates an isometric group. Again, by Theorem 2.2, there is a unique bounded operator $X_2 : F \to ker(P)$ such that

$$-A_2X_2 + X_2B = -C_2, \text{ or } A_2X_2 - X_2B = C_2. \tag{2.8}$$

Let now $X = X_1 + X_2$, then $AX - XB = A(X_1 + X_2) - (X_1 + X_2)B = (A_1X_1 - X_1B) + (A_2X_2 - X_2B) = C_1 + C_2 = C$. Thus, X is a bounded solution of (2.1). The uniqueness of X follows from the fact that PX and $(I - P)X$ are unique bounded solutions of (2.7) and (2.8), respectively. ♣

In the following we apply above results to study the existence and regularity of solutions of Cauchy problems.

Corollary 2.4 *Let A be the generator of a C_0-semigroup $(T(t))$ on E such that $\omega(T) < 0$ and $\mathcal{D} : W^{p,1}(\mathrm{I\!R}, E) \to L^p(\mathrm{I\!R}, E)$ be given by $\mathcal{D} = d/dt$. Then the equation*

$$AX - X\mathcal{D} = \delta_0 \tag{2.9}$$

has a unique solution.

Proof. It is well-known that δ_0 is $\mathcal{D}$-bounded and $-\mathcal{D}$ is generator of the shift C_0-group $S(t)$ given by $S(t)f(s) = f(s-t)$. Hence, $\omega(S) = 0$. Moreover,

$$\begin{aligned} \|R(t)f\| = \|\int_0^t T(s)\delta_0 S(s) f ds\| &= \|\int_0^t T(s)f(-s)ds\| \\ &\leq M\int_0^t \|f(-s)\|ds \leq M\|f\|. \end{aligned}$$

Hence $R(t)$, $t \geq 0$, is uniformly bounded. Thus, by Theorem 2.2, (2.9) has a unique bounded solution. ♣

From Corollary 2.4 we obtain

Corollary 2.5 *Let $p \geq 1$ and A be the generator of a C_0-semigroup in E. Then the operator*

$$\mathcal{A} := \begin{pmatrix} A & \delta_0 \\ 0 & \mathcal{D} \end{pmatrix}$$

with $D(\mathcal{D}) := D(A) \times W^{1,1}(\mathrm{I\!R}, E)$ is the generator of a C_0-semigroup on $E \times L^p(\mathrm{I\!R}, E)$.

Proof. Without loss of generality, we assume $\omega(T) < 0$. Then, by Corollary 2.4, Equation (2.9) has a unique solution. Hence, $\mathcal{A}$ is similar to the generator $\begin{pmatrix} A & 0 \\ 0 & \mathcal{D} \end{pmatrix}$ and thus is a generator. ♣

It is well-known (see e.g. [7]) that if $(u_1, u_2)^T$ is a (classical) solution of the Cauchy problem

$$\begin{cases} \mathcal{U}'(t) = \mathcal{A}\mathcal{U}(t) \quad t \geq 0 \\ \mathcal{U}(0) = (u_0, f)^T \end{cases} \tag{2.10}$$

on $E \times F(\mathbb{R}, E)$, where $F(\mathbb{R}, E)$ is a function space, then the first component u_1 is the (classical) solution of the inhomogeneous Cauchy problem

$$\begin{cases} u'(t) = Au(t) + f(t) & t \geq 0 \\ u(0) = u_0. \end{cases} \tag{2.11}$$

From the above observation and Corollary 2.5 we obtain the following.

Corollary 2.6 *Let A be the generator of a C_0-semigroup and $f \in W^{1,1}(\mathbb{R}, E)$, then* (2.11) *has a unique classical solution.*

Let us now recall the definition of *extrapolation space*. Let A be the generator of a C_0-semigroup $(T(t))$ on a Banach space E and $\lambda \in \varrho(A)$. On E we introduce a new norm by

$$\|x\|_{-1} = \|R(\lambda, A)x\|.$$

Then the completion of $(E, \|\cdot\|_{-1})$ is called the *extrapolation space* of E associated with A, and is denoted by E_{-1}. It is shown that the operator $T(t)$ can be uniquely extended to a bounded operator on the Banach space E_{-1}. The result is a C_0-semigroup on E_{-1}, denoted by $(T_{-1}(t))$. The semigroup $(T_{-1}(t))$ is called the *extrapolated semigroup* of $(T(t))$.If we denote by A_{-1} the generator of $(T_{-1}(t))$ on E_{-1}, then we have the following properties (see more details in [3], [7]).

(*i*) $\|T_{-1}(t)\|_{L(E_{-1})} = \|T(t)\|_{L(E)}$;

(*ii*) E is dense in E_{-1} and $D(A_{-1}) = E$;

(*iii*) $A_{-1} : E \to E_{-1}$ is the unique extension of $A : D(A) \to E$ to $E \to E_{-1}$.

The following two corollaries show the existence and uniqueness of classical solution of nonhomogenous Cauchy problem (2.11) for the case that the nonhomogenous term f is not differentiable. Since their proofs are similar, we present here only one of them.

Corollary 2.7 *Let A be the generator of an analytic semigroup. Then* (2.11) *has a unique classical solution for every $x \in D(A)$ and Hölder continuous function $f \in H_\alpha(\mathbb{R}, E)$.*

Corollary 2.8 *Let A be the generator of a C_0-semigroup on E. Then* (2.11) *has a unique classical solution for every $x \in D(A)$ and $f \in BUC(\mathbb{R}, [D(A)])$, where $[D(A)]$ is the Banach space $(D(A), \|\cdot\|_A)$ with the norm $\|x\|_A = \|x\| + \|Ax\|$.*

Proof of Corollary 2.8. Without loss of generality we assume $\omega(A) < 0$. In view of the previous observation we have only to show that $\mathcal{A} := \begin{pmatrix} A & \delta_0 \\ 0 & \mathcal{D} \end{pmatrix}$ with $D(\mathcal{A}) = D(A) \times BUC(\mathbb{R}, [D(A)])$ is generator of a C_0-semigroup on $E \times BUC_{-1}\mathbb{R}, [D(A)])$, where $BUC_{-1}(\mathbb{R}, [D(A)])$ is the extrapolated space of $BUC(\mathbb{R}, [D(A)])$ associated with $\mathcal{D} = d/dt$ on $BUC(\mathbb{R}, [D(A)])$. This is done if we show that there is a bounded solution of operator equation

$$AX - X\mathcal{D} = \delta_0, \tag{2.12}$$

where $F := BUC_{-1}(\mathbb{R}, [D(A)])$. It is easy to see that δ_0 is $\mathcal{D}$-bounded. Since $\omega(\mathcal{D}) = 0$ we have $\omega(A) + \omega(\mathcal{D}) < 0$. Moreover, let $f \in BUC(\mathbb{R}, [D(A)])$ and $g(t) = R(1, \mathcal{D})f(t)$, then we have

$$\begin{aligned}
\|R(t)f\| &= \|\int_0^t T(s)f(-s)ds\| \\
&= \|\int_0^t T(s)(g(-s) - g'(-s))ds\| \\
&\square \int_0^t T(s)g(-s)ds\| + \|\int_0^t T(s)g'(-s)ds\| \\
&\square \|\int_0^t T(s)g(-s)ds\| + \|\int_0^t T(s)Ag(-s)ds\| + \|T(t)g(-t)\| + \|g(0)\| \\
&\square C(\sup_{s\in\mathbb{R}} \|g(s)\| + \sup_{s\in\mathbb{R}} \|Ag(s)\|) \\
&= C\|g\|_{BUC(\mathbb{R},[D(A)])} \\
&= C\|R(1, \mathcal{D})f\|_{BUC(\mathbb{R},[D(A)])} \\
&= C\|f\|_{BUC_{-1}(\mathbb{R},[D(A)])}
\end{aligned}$$

Here we used the fact $\int_0^t T(s)g'(-s)ds = \int_0^t AT(s)g(-s)ds - T(t)g(-t) + g(0)$. Since $BUC(\mathbb{R}, [D(A)])$ is dense in $BUC_{-1}(\mathbb{R}, [D(A)])$, $R(t)$ is uniformly bounded. By Theorem 2.2, (2.12) has a unique bounded solution, and this concludes the proof. ♣

We complete this section with the following result, which is very helpful for studying properties of unbounded pertubed generators.

Theorem 2.9 *For the operator matrix* $\mathcal{A} = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ *with* $D(\mathcal{A}) = D(A) \times D(B)$ *we assume that* A *is the generator of a* C_0*-semigroup on* E*,* B *is the generator of a bounded* C_0*-group on* F *and* C *is* B*-bounded. Then* $\mathcal{A}$ *is a generator of a* C_0*-semigroup on* $E \times F$ *if and only if* $\mathcal{A}$ *is of the form*

$$\mathcal{A} = \mathcal{Q} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mathcal{Q}^{-1} + \mathcal{L}$$

with an isomorphism $\mathcal{Q}$ *and a bounded operator* $\mathcal{L}$ *in* $E \times F$*.*

Proof. We have only to show the "only if" part. By the assumption we have $\omega(S) = 0$, where $(S(t))_{t\in\mathbb{R}}$ is the group of $-B$. Without loss of generality we assume $\omega(T) < 0$. Since $\mathcal{A}$ is generator of a C_0-semigroup on $E \times F$, by [6, Theorem 3.1], we have

$$V(t) := \int_0^t T(t-s)CS(-s)ds = \int_0^t T(s)CS(s-t)ds$$

is bounded for $t \geq 0$. Thus,

$$\begin{aligned} \|R(t)f\| &= \| \int_0^t T(s)CS(s)fds\| \\ &= \| \int_0^t T(s)CS(s-t)S(t)fds\| \\ &\square \ \|V(t)\|.\|S(t)f\| \ \square \ \|V(t)\|.\|S(t)\|.\|f\|. \end{aligned}$$

Hence $R(t)$ is bounded for every $t \geq 0$. By Corollary 2.2, the equation $AX - XB = C$ has a unique bounded solution X. Therefore,

$$\mathcal{A} = \begin{pmatrix} I & -X \\ 0 & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & X \\ 0 & I \end{pmatrix},$$

and the corollary is proved. ♣

3 Regularity of solutions of differential equations

In this section we consider the differential equation on the line $\mathbb{R}$

$$u'(t) = Au(t) + f(t), \ t \in \mathbb{R}, \tag{3.1}$$

where A is generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on E and $f \in L^p(\mathbb{R}, E)$. We say that the continuous function $u(t)$ is *a mild solution* of (3.1), if

$$u(t) = T(t-s)u(s) + \int_s^t T(t-\tau)f(\tau)d\tau \tag{3.2}$$

for all $t \geq s$. It turns out that the existance and uniqueness of bounded mild solutions of (3.1) is closely related to the solvability of operator equations, as the following theorem shows.

Theorem 3.1 *Let A be a generator of a C_0-semigroup $(T(t))_{t\geq 0}$ with $\sigma(A) \cap i\mathbb{R} = \emptyset$. Then the following are equivalent.*

(a) For each function f in $L^p(\mathbb{R}, E)$ there exists a unique mild solution of (3.1), which is bounded.

(b) There exists a unique bounded solution of operator equation

$$AX - X\mathcal{D} = -\delta_0, \tag{3.3}$$

where $F = L^p(\mathbb{R}, E)$, $\mathcal{D} : W^{p,1}(\mathbb{R}, E) \to F$ and $\delta_0 : W^{p,1}(\mathbb{R}, E) \to E$.

Proof. $(i) \Rightarrow (ii)$. Let$C(\mathbb{R}, E)$ be the space of all bounded continuous functions over E and $G : L^p(\mathbb{R}, E) \to C(\mathbb{R}, E)$ be the operator defined by $Gf = u$ where u is the unique mild solution of (3.1) corresponding to f. By standard argument, it is easy to see that G is a closed, and hence, a bounded operator. Define $Xf = (Gf)(0)$. Then$X$ is a bounded operator from $L^p(\mathbb{R}, E)$ to E.

Let now $f \in W^{p,1}(\mathbb{R}, E)$. By Lemma 2.6, $u = Gf$ is a classical solution of (3.1), i.e.,

$$(Gf)'(t) = A(Gf)(t) + f(t). \tag{3.4}$$

Note that, $(Gf)' = G(f')$. Hence $(Gf')(t) = A(Gf)(t) + f(t)$. Taking $t = 0$, we have $AXf - X\mathcal{D}f = -\delta_0 f$ for $f \in \mathcal{D}$, i.e. X is a bounded solution of (3.3).

To show the uniqueness, we assume that X_0 is a solution of equation $AX - X\mathcal{D} = 0$. Then for every $f \in \mathcal{D}$ the function $u \in \mathcal{M}$ defined by $u(t) = XS(t)f$ is a classical solution of Equation (3.1), since

$$u'(t) = X\mathcal{D}S(t)f = (AX + \delta_0)S(t)f = Au(t) + f(t)$$

for all $t \in \mathbb{R}$.

Let now $f \in \mathcal{M}$ and $(f_k)_{k\in\mathbb{N}} \subseteq D(\mathcal{D}$ with $\lim_k f_k = f$. Then $\lim_k u_k = \lim_k XS(\cdot)f_k = XS(\cdot)f$. Hence, taking the limit on both sides of $u_k = Gf_k$ as $k \to \infty$ we get $XS(\cdot)f = Gf$, i.e., $u = XS(\cdot)f$ is a mild solution of (3.1). Assume now that X_1 and X_2 are two solutions of (3.3). Then, for every $f \in \mathcal{M}$, $u = (X_1 - X_2)S(\cdot)f$ is a mild solution Equation (3.1). By the uniqueness of the mild solution we have $u \equiv 0$, which implies $X_1 = X_2$.

$(ii) \Rightarrow (i)$ We have shown above that, if X is a bounded solution of (3.3), then $u(t) := XS(t)f$ is a mild solution of Equation (3.1). It remains to be shown that this solution is unique. In order to do this, assume that v is a mild solution of the homogeneous equation $u'(t) = Au(t)$, $t \in \mathbb{R}$. It is the well-known Tauberian Theorem (see e.g. [14]) that the spectrum of function f, $sp(f)$, satisfies $isp(v) \subseteq \sigma(A)$. By assumption, $\sigma(A) \cap i\mathbb{R} = \emptyset$, so that $sp(v) = \emptyset$, and so $v \equiv 0$ (see [11, p. 22]), and the theorem is proved. ♣

From Lemma 2.4 and Theorem 3.1 we obtain

Corollary 3.2 *If A is the generator of an exponentially dichotomic C_0-semigroup, then for every function f if $L^p(\mathbb{R}, E)$, Equation* (3.1) *has a unique bounded mild solution.*

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