On the Periodic Mild Solutions to Complete Higher Order Differential Equations on Banach Space. [§]

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Abstract

For the complete higher order differential equation

$$u^{(n)}(t) = \sum_{k=0}^{n-1} A_k u^{(k)}(t) + f(t), \quad 0 \le t \le T,$$

on a Banach space E, we give necessary and sufficient conditions for the periodicity of mild solutions. The results, which are proved in a simple manner, generalize some well-known ones.

1 Introduction

In this paper we are concerned with the periodicity of solutions of the complete higher order differential equation:

$$u^{(n)}(t) = \sum_{j=0}^{n-1} A_j u^{(j)}(t) + f(t), \quad 0 \le t \le T, \tag{1.1}$$

where A_j are linear, closed operators on a Banach space E and f is a function from $[0, T]$ to E.

The asymptotic behavior and, in particular, the periodicity of solutions of the higher order differential equation

$$u^{(n)}(t) = Au(t) + f(t), \quad 0 \le t \le T, \tag{1.2}$$

has been an subject of intensive study for recent decades. When $n = 1$, it is well-known [7] that, if A is an $n \times n$ matrix on $\mathbb{C}^n$, then (1.2) admits a unique

[§]2000 AMS Subject Classification: Primary 34 G 10, 34 K 06, Secondary 47 D 06.
Keywords and phrases: Abstract higher order differential equations, Fourier series, periodic mild solutions, operator semigroups and cosine families.

T-periodic solution for each continuous T-periodic forcing term f if and only if $\lambda_k = 2k\pi t/T$, $k \in \mathbb{Z}$, are not eigen-values of A. That result was extended by Krein and Dalecki [4] to the Cauchy problem in an abstract Banach space. It was claimed [4, Theorem II 4.3] that, if A is a linear, bounded operator on E, then (1.2) admits a unique T-periodic solution for each $f \in C[0,T]$ if and only if $2k\pi i/T \in \varrho(A)$, $k \in \mathbb{Z}$. Here $\varrho(A)$ denotes the resolvent set of A. For unbounded operator A, the situation changes dramatically and the above statement generally fails. When A generates a strongly continuous semigroup, periodicity of solutions of (1.2) has intensively been studied recently (see e.g. [9, 10, 14, 15]). Corresponding results on the periodic solutions of the second order differential equation were obtained in [3, 17], when A is generator of a cosine family. Related results on the periodicity of solutions of (1.2), when A is a closed operator, can be found in [5, 8, 12, 13, 16] and the references therein.

Unfortunately, for the complete higher differential equations, we have little consideration about the regularity of their solutions, mainly because of the complexity of the structure of the equation. In this paper we investigate the periodicity of mild solutions of the complete higher order differential equation (1.1) when A_j, $j = 0, 1, ..., n-1$, are linear, closed operators. The main tool we use here is the Fourier series method. For an integrable function $f(t)$ from $[0,T]$ to E, the Fourier coefficient of $f(t)$ is defined as

$$f_k = \frac{1}{T}\int_0^T f(s)e^{-2k\pi is/T}ds, \ k \in Z.$$

Then $f(t)$ can be represented by Fourier series

$$f(t) \approx \sum_{k=-\infty}^{\infty} e^{2k\pi it/T} f_k.$$

First, we give a general definition of mild solution to the complete higher order differential equation (1.1). This definition is an extension of that introduced in [2, 13], when the equation has the form of (1.2). We then establish the relationship between the Fourier coefficients of the periodic solutions of (1.1) and those of the inhomogeneity f. As the main result, we give different equivalent conditions so that (1.1) admits a unique periodic solution for each inhomogeneity f in a certain function space. Our result generalizes some well-known ones, as in Section 3 we present several particular cases, among which, A generates a C_0 semigroup and a cosine family.

Let us fix some notations. A continuous function on $[0, T]$ is said to be T-periodic if $u(0) = u(T)$. For the sake of simplicity (and without loss of generality) we assume $T = 1$ and put $J := [0, 1]$. For $p \geq 1$, $L_p(J)$ denotes the space of E-valued integrable functions on J with $\|f\|_{L_p(J)} = \int_0^1 \|f(t)\|^p dt < \infty$ and $C(J)$ the space of continuous functions on J with and $\|f\|_{C(J)} = \sup_J \|f(t)\| < \infty$. Moreover, for $m > 0$ we define the following function spaces:

1) $W_p^m(J) := \{f \in L_p(J) : f', f'', ..., f^{(m)} \in L_p(J)\}$. $W_p^m(J)$ is then a Banach space with the norm

$$\|f\|_{W_p^m} := \sum_{j=0}^{m} \|f^{(j)}\|_{L_p(J)}.$$

2) $P^m(J) := \{f \in C(J) : f, f', ..., f^{(m)} \text{ are in } P(J)\}$. That means $P^m(J)$ is the space of all functions on J, which can be extended to 1-periodic, m-times continuously differentiable functions on $\mathbb{R}$. $P^m(J)$ is a Banach space with the norm

$$\|f\|_{P^m(J)} := \sum_{j=0}^{m} \|f^{(j)}\|_{C(J)}.$$

3) $WP_p^m(J) := P^{m-1}(J) \cap W_p^m(J)$. It is easy to see that $WP_p^m(J)$ is a Banach space with $W_p^m(J)$-norm.

We will use the following simple lemma.

Lemma 1.1 *If F is a continuous function on J such that $f = F' \in L_p(J)$, then for $k \neq 0$ we have*

$$F_k = \frac{1}{2k\pi i} f_k + \frac{F(0) - F(1)}{2k\pi i},$$

where f_k and F_k are the Fourier series of f and F, respectively.

Throughout this paper, if not otherwise indicated, we assume that A_i, $i = 0, 1, ..., n-1$, are linear, closed and densely defined operators on E with $\cap_{j=0}^{n-1} D(A_j)$ dense in E, that satisfy the following condition:

Condition F: *There exists a linear, closed operator B on E with $0 \in \varrho(B)$ such that $B^{-1}A_j$ can be extended to bounded operators $B_j = \overline{B^{-1}A_j}$ for all $j = 0, 1, ..., n-1$.*

For a number $\lambda \in \mathbb{C}$, define the operator $S(\lambda)$ by

$$\begin{aligned} S(\lambda) &:= \lambda^n - B(\sum_{j=0}^{n-1} \lambda^j \overline{B^{-1}A_j}) \\ &= \lambda^n - B(\sum_{j=0}^{n-1} \lambda^j B_j) \end{aligned}$$

with

$$D(S(\lambda)) = \{x \in E : \sum_{j=0}^{n-1} \lambda^j B_j x \in D(B)\}.$$

It is not hard to see that $\cap_{j=0}^{n-1} D(A_J) \subseteq D(S(\lambda)$. Moreover, since $B^{-1}S(\lambda)$ are bounded, $S(\lambda)$ are closed operators. Finally, we define the resolvent $\varrho(S)$ by

$$\varrho(S) := \{\lambda \in \mathbb{C} : S(\lambda) \text{ is injective and surjective}\}$$

and the spectrum $\sigma(S) := \mathbb{C} \setminus \varrho(S)$. Since $S(\lambda)$ are closed operators, if $\lambda \in \varrho(S)$, then $S(\lambda)^{-1}$ is a bounded operator on E.

2 Periodic Mild Solutions of Higher Order Differential Equations

Let F be either $L_p(J)$ or $C(J)$ space. We define the operator $I : F \to C(J)$ by $If(t) := \int_0^t f(s)ds$ and $I^j f := I(I^{j-1}f)$.

Definition 2.1 *(1) Suppose $f \in L_p(J)$. A continuous function u is called a mild solution of* (1.1) *on J, if $\sum_{j=0}^{n-1} B_j I^{n-j} u(t) \in D(B)$ and there are* *vector* x_0, x_1, ..., x_{n-1} *in* E *such that*

$$u(t) = \sum_{j=0}^{n-1} \frac{t^i}{j!} x_j + B\Big(\sum_{j=0}^{n-1} B_j I^{n-j} u(t)\Big) + I^n f(t) \tag{2.1}$$

for all $t \in J$.

(2) *Suppose* f *is a continuous function on* J. *A function* u *is a classical solution of* (1.1) *on* J, *if* u *is* n*-times continuously differentiable,* $\sum_{j=0}^{n-1} B_j u^{(j)} u(t) \in D(B)$ *and*

$$u^{(n)}(t) = B\Big(\sum_{j=0}^{n-1} B_j u^{(j)}(t)\Big) + f(t)$$

holds for $t \in J$.

The mild solution to (1.1) defined by (2.1) is really an extension of classical solution in the sense that every classical solution is a mild solution and conversely, if a mild solution is n-times continuously differentiable, then it is a classical solution. That statement is actually contained in the following lemma. For the sake of simplicity, for $j < 0$, we denote $I^j u(t) := u^{(j)}(t)$.

Lemma 2.2 *Suppose* $0 \leq m \leq n$ *and* u *is a mild solution of* (1.1), *which is* m*-times continuously differentiable. Then we have* $\sum_{j=0}^{n-1} B_j I^{n-m-j} u(t) \in D(B)$ *and*

$$u^{(m)}(t) = \sum_{j=m}^{n-1} \frac{t^{j-m}}{(j-m)!} x_j + B\Big(\sum_{j=0}^{n-1} B_j I^{n-m-j} u(t)\Big) + I^{n-m} f(t). \tag{2.2}$$

Proof. If $m = 0$, then (2.2) coincides with (2.1). We prove for $m = 1$: Let

$$v(t) := B\Big(\sum_{j=0}^{n-1} B_j I^{n-j} u(t)\Big) = u(t) - \sum_{j=0}^{n-1} \frac{t^j}{j!} x_j - I^n f(t).$$

Then, by the assumptions, v is continuously differentiable and

$$v'(t) = u'(t) - \sum_{j=1}^{n-1} \frac{t^{j-1}}{(j-1)!} x_j - I^{n-1} f(t).$$

Let $h > 0$ and put

$$v_h := \sum_{j=0}^{n-1} B_j \frac{1}{h} \int_t^{t+h} I^{n-j-1} u(s) ds.$$

Then $v_h \to \sum_{j=0}^{n-1} B_j (I^{n-j-1} u)(t)$ for $h \to 0$ and

$$\begin{aligned} Bv_h &= B \sum_{j=0}^{n-1} \frac{1}{h} \Big(B_j \int_0^{t+h} I^{n-j-1} u(s) ds - B_j \int_0^t I^{n-j-1} u(s) ds \Big) \\ &= \frac{1}{h} B \sum_{j=0}^{n-1} B_j \int_0^{t+h} I^{n-j-1} u(s) ds - \frac{1}{h} B \sum_{j=0}^{n-1} B_j \int_0^t I^{n-j-1} u(s) ds \\ &= \frac{1}{h} (v(t+h) - v(t)) \\ &\to v'(t) \text{ for } h \to 0. \end{aligned}$$

Since B is a closed operator, we obtain that $\sum_{j=0}^{n-1} B_j (I^{n-j-1} u)(t) \in D(B)$ and

$$B \sum_{j=0}^{n-1} B_j (I^{n-j-1} u)(t) = u'(t) - \sum_{j=1}^{n-1} \frac{t^{j-1}}{(j-1)!} x_j - I^{n-1} f(t),$$

from which (2.2) with $m = 1$ follows. If $m > 1$, we obtain (2.2) by repeating the above process $(m-1)$ times. ♣

In particular, if f is continuous and the mild solution u is n-times continuously differentiable, i.e. $m = n$, then (2.2) becomes $u^{(n)}(t) = B \sum_{j=0}^{n-1} B_j I^{-j} u(t) + f(t) = B \sum_{j=0}^{n-1} B_j u^{(j)}(t) + f(t)$, i.e. u is a classical solution of (1.1).

We now consider the mild solutions which are $(n-1)$ times continuously differentiable. The following proposition describes the connection between the Fourier coefficients of those solutions and those of $f(t)$. Before stating the proposition, we define the bounded operator on E:

$$S'(\lambda) := \lambda^n B^{-1} - \sum_{j=0}^{n-1} \lambda^j B_j.$$

It is not hard to see that $S'(\lambda)$ is the bounded extension of $B^{-1} S(\lambda)$.

Proposition 2.3 *Suppose $f \in L_p(J)$ and u is a mild solution of* (1.1)*, which is $(n-1)$ times continuously differentiable. Then*

$$S'(2k\pi i)u_k - B^{-1}f_k \;=\; \sum_{j=0}^{n-1}\Big((2k\pi i)^{n-j-1}B^{-1} - \sum_{m=j+1}^{n-1}(2k\pi i)^{m-j-1}B_m\Big)[u^{(j)}(0) - u^{(j)}(1)] \tag{2.3}$$

for $k \in \mathbb{Z}$.

Proof. Denote $x_m := u^{(m)}(0) - u^{(m)}(1)$. Let $u_k^{(m)}$ be the k^{th} Fourier coefficient of $u^{(m)}$. Using the identity

$$u_k^{(m)} = [u^{(m-1)}(1) - u^{(m-1)}(0)] + (2k\pi i)u_k^{(m-1)} = -x_{m-1} + (2k\pi i)u_k^{(m-1)} \tag{2.4}$$

for $m = 0, 1, 2, ..., n-1$ (by Lemma 1.1), we obtain

$$u_k^{(m)} = -\sum_{j=0}^{m-1}(2k\pi i)^{m-j-1}x_j + (2k\pi i)^m u_k. \tag{2.5}$$

Let u now be a mild solution of (1.1), which is $(n-1)$ times continuously differentiable. By Lemma 2.2, it satisfies the following identity

$$u^{(n-1)}(t) = u^{(n-1)}(0) + B\Big(\sum_{m=1}^{n-1}B_m u^{(m-1)}(t) + B_0\int_0^t u(s)ds\Big) + \int_0^t f(s)ds. \tag{2.6}$$

First, if $k = 0$, then using (2.6) with $t = 0$ and $t = 1$ we have

$$0 = \sum_{m=1}^{n-1}B_m u^{(m-1)}(0)$$

and

$$-x_{n-1} = B\Big(\sum_{m=1}^{n-1}B_m u^{(m-1)}(1) + B_0 u_0\Big) + f_0,$$

from which we have

$$\begin{aligned} -B^{-1}x_{n-1} - B^{-1}f_0 \;&=\; \sum_{m=1}^{n-1}B_m u^{(m-1)}(1) + B_0 u_0 \\ &=\; -\sum_{m=1}^{n-1}B_m x_{m-1} + B_0 u_0, \end{aligned} \tag{2.7}$$

which means that (2.3) holds for $k = 0$.

Next, if $k \neq 0$, taking the k^{th} Fourier coefficient on both sides of (2.6), we obtain

$$\begin{aligned} u_k^{(n-1)} &= B\Big(\sum_{m=1}^{n-1} B_m u_k^{(m-1)} + B_0 \int_0^1 e^{-2k\pi is} \int_0^s u(\tau)d\tau ds\Big) + \int_0^1 e^{-2k\pi is} \int_0^s f(\tau)d\tau ds \\ &= B\Big(\sum_{m=1}^{n-1} B_m u_k^{(m-1)} + \frac{B_0 u_k - B_0 u_0}{2k\pi i}\Big) + \frac{f_k - f_0}{2k\pi i}, \end{aligned}$$

from which we obtain

$$\sum_{m=1}^{n-1} B_m u_k^{(m-1)} + \frac{B_0 u_k - B_0 u_0}{2k\pi i} = B^{-1} u_k^{(n-1)} + \frac{B^{-1} f_0 - B^{-1} f_k}{2k\pi i} \tag{2.8}$$

Here we used Lemma 1.1 for $F(t) = \int_0^t u(\tau)d\tau ds$ and $F(t) = \int_0^t f(\tau)d\tau ds$. Using (2.5) for both sides of (2.8) we have

$$\begin{aligned} \textstyle\sum_{m=1}^{n-1} \quad & B_m\Big(\sum_{j=0}^{m-2} -(2k\pi i)^{m-j-2} x_j + ((2k\pi i)^{m-1} u_k\Big) + \frac{B_0 u_k - B_0 u_0}{2k\pi i} \\ = \quad & B^{-1}(\sum_{j=0}^{n-2} -(2k\pi i)^{n-j-2} x_j + (2k\pi i)^{n-1} u_k) + \frac{B^{-1} f_0 - B^{-1} f_k}{2k\pi i}, \end{aligned}$$

from which it implies

$$\begin{aligned} & S'(2k\pi)u_k - B^{-1} f_k = \\ = \ & \sum_{j=0}^{n-2} (2k\pi i)^{n-j-1} B^{-1} x_j - \sum_{m=1}^{n-1} B_m\Big(\sum_{j=0}^{m-2} (2k\pi i)^{m-j-1} x_j\Big) - (B_0 u_0 + B^{-1} f_0). \end{aligned} \tag{2.9}$$

Using Identity (2.7) for (2.9) we have

$$\begin{aligned} & S'(2k\pi)u_k - B^{-1} f_k = \\ = \ & \sum_{j=0}^{n-2} (2k\pi i)^{n-j-1} B^{-1} x_j - \sum_{m=1}^{n-1} B_m\Big(\sum_{j=0}^{m-2} (2k\pi i)^{m-j-1} x_j\Big) - (-B^{-1} x_{n-1} + \sum_{m=1}^{n-1} B_m x_{m-1}) \\ = \ & B^{-1} x_{n-1} + \sum_{j=0}^{n-2} (2k\pi i)^{n-j-1} B^{-1} x_j - \sum_{m=1}^{n-1} B_m\Big(\sum_{j=0}^{m-2} (2k\pi i)^{m-j-1} x_j + x_{m-1}\Big) \end{aligned}$$

$$
\begin{aligned}
&= B^{-1}x_{n-1} + \sum_{j=0}^{n-2}(2k\pi i)^{n-j-1}B^{-1}x_j - \sum_{m=1}^{n-1}\sum_{j=0}^{m-1}(2k\pi i)^{m-j-1}B_m x_j \\
&= \sum_{j=0}^{n-1}(2k\pi i)^{n-j-1}x_j - \sum_{j=0}^{n-2}\sum_{m=j+1}^{n-1}(2k\pi i)^{m-j-1}A_m x_j \\
&= B^{-1}x_{n-1} + \sum_{j=0}^{n-2}\Big((2k\pi i)^{n-j-1}B^{-1} - \sum_{m=j+1}^{n-1}(2k\pi i)^{m-j-1}B_m\Big)x_j \\
&= \sum_{j=0}^{n-1}\Big((2k\pi i)^{n-j-1}B^{-1} - \sum_{m=j+1}^{n-1}(2k\pi i)^{m-j-1}B_m\Big)x_j,
\end{aligned}
$$

from which (2.3) follows. ♣

The interesting point of Proposition 2.3 is that the Fourier coefficients of the mild solution u depend not only on u but also on its derivatives. If u is a periodic solution, then we have a nice relationship between Fourier coefficients of u and those of f, as the following proposition shows.

Proposition 2.4 *Suppose $f \in L_p(J)$ and u is a mild solution of (1.1), which is $(n-1)$ times continuously differentiable. Then u is 1-periodic if and only if*

$$S(2k\pi i)u_k = f_k \tag{2.10}$$

for every $k \in \mathbb{Z}$.

First we prove the following lemma.

Lemma 2.5 *Suppose x_0, x_1, ..., x_{n-1} are n vectors in E. Then, from the identities*

$$\sum_{j=0}^{n-1}\Big((2k\pi i)^{n-j-1}B^{-1} - \sum_{m=j+1}^{n-1}(2k\pi i)^{m-j-1}B_m\Big)x_j = 0 \tag{2.11}$$

for all $k \in \mathbb{Z}$ we have $x_0 = x_1 = ... = x_{n-1} = 0$.

Proof. We first show $x_0 = 0$. Put

$$d := \max\{\|A_m x_j\|, \|B^{-1}x_j\| : 1 \le m \le (n-1);\ 0 \le j \le (n-1)\}$$

For the sake of simplicity, denote $\alpha := 2k\pi i$. From (2.11) we have

$$\|\alpha^{n-1}B^{-1}x_0 - \sum_{m=1}^{n-1} \alpha^{m-1}B_m x_0\| = \|\sum_{j=1}^{n-1}\Big(\alpha^{n-j-1}B^{-1} - \sum_{m=j+1}^{n-1}\alpha^{m-j-1}B_m\Big)x_j\| \tag{2.12}$$

Using triangle inequality for each side of (2.12) we have

$$\begin{aligned}\|\alpha^{n-1}B^{-1}x_0 - \sum_{m=1}^{n-1} \alpha^{m-1}B_m x_0\| &\geq |\alpha|^{n-1}\|B^{-1}x_0\| - d\sum_{m=1}^{n-1}|\alpha|^{m-1} \\ &= |\alpha|^{n-1}\|B^{-1}x_0\| - d\frac{|\alpha|^{n-1}-1}{|\alpha|-1}\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}\|\sum_{j=1}^{n-1}\Big(\alpha^{n-j-1}B^{-1} - \sum_{m=j+1}^{n-1}\alpha^{m-j-1}B_m\Big)x_j\| &\leq d\sum_{j=1}^{n-1}\Big(|\alpha|^{n-j-1} + \sum_{m=j+1}^{n-1}|\alpha|^{m-j-1}\Big) \\ &= d\sum_{j=1}^{n-1}\sum_{m=j+1}^{n}|\alpha|^{m-j-1} \\ &= d\sum_{j=1}^{n-1}\frac{|\alpha|^{n-j}-1}{|\alpha|-1} \\ &= d\frac{\sum_{j=1}^{n-1}|\alpha|^{n-j}-(n-1)}{|\alpha|-1} \\ &= d\frac{(|\alpha|^n-|\alpha|)/(|\alpha|-1)-(n-1)}{|\alpha|-1} \\ &= d\frac{|\alpha|^n - n|\alpha| + (n-1)}{(|\alpha|-1)^2}.\end{aligned} \tag{2.14}$$

Combining (2.13) and (2.14) we obtain

$$|\alpha|^{n-1}\|B^{-1}x_0\| \leq d\Big(\frac{|\alpha|^{n-1}-1}{|\alpha|-1} + \frac{|\alpha|^n - n|\alpha| + (n-1)}{(|\alpha|-1)^2}\Big),$$

which implies

$$\|B^{-1}x_0\| \leq d\frac{2|\alpha|^n - |\alpha|^{n-1} - (n+1)|\alpha| + n}{|\alpha|^{n+1} - 2|\alpha|^n + |\alpha|^{n-1}}.$$

Replace $\alpha = 2k\pi i$ back, we have

$$\|B^{-1}x_0\| \le d\frac{2|2k\pi|^n - |2k\pi|^{n-1} - (n+1)|2k\pi| + n}{|2k\pi|^{n+1} - 2|2k\pi|^n + |2k\pi|^{n-1}} \tag{2.15}$$

for all $k \in \mathbb{Z}$. Let $k \to \infty$, then the right hand side of (2.15) approaches to zero. Hence $B^{-1}x_0 = 0$, and thus, $x_0 = 0$. With the same manner, we can show $x_1 = 0$, and then $x_2 = 0$ and so on, and the lemma is proved. ♣

Proof of Proposition 2.4. Suppose u is a mild 1-periodic solution of (1.1), which is $(n-1)$ times continuously differentiable, then u', u'', ..., $u^{(n-1)}$ are also 1-periodic, i.e. $u(1) = u(0)$, $u'(1) = u'(0)$, ..., $u^{(n-1)}(1) = u^{(n-1)}(0)$. Hence (2.10) follows directly from (2.3). Conversely, suppose (2.10) holds for all $k \in \mathbb{Z}$. That means

$$\sum_{j=0}^{n-1}\Big((2k\pi i)^{n-j-1}B^{-1} - \sum_{m=j+1}^{n-1}(2k\pi i)^{m-j-1}B_m\Big)[u^{(j)}(0) - u^{(j)}(1)] = 0$$

for all $k \in \mathbb{Z}$. By Lemma 2.5 then $u^{(j)}(0) - u^{(j)}(1) = 0$ for all $j = 0, 1, ..., n-1$. In particular, u is 1-periodic function. ♣

From Proposition 2.4 we obtain

Corollary 2.6 *Suppose $f \in L_p(J)$. Then*

(i) If $S(2k\pi i)$ is injective for $k \in \mathbb{Z}$, then Equation (1.1) *has at most one 1-periodic mild solution, which belongs to $P^{n-1}(J)$.*

(ii) If there exists a number $k \in \mathbb{Z}$ such that $f_k \notin RangeS(2k\pi i)$, then Equation (1.1) *has no periodic mild solution which belongs to $P^{n-1}(J)$.*

We now are going to find conditions such that for each function $f \in W_p^m(J)$, Equation (1.1) has a unique 1-periodic mild solution, which is $(n-1)$ times continuously differentiable. We are in the position to state the main result.

Theorem 2.7 *Let A_j, $j = 0, 2, ..., n-1$, be linear, closed operators on E. The following statements are equivalent.*

(i) For each $f \in W_p^m(J)$, *Equation* (1.1) *admits a 1-periodic mild solution, which belong to* $W_p^n(J)$ *(i.e.* $u \in PW^{n-1}(J)$*);*

(ii) For each $k \in \mathbb{Z}$, $2k\pi i \in \varrho(S)$ *and there exists a constant* $C > 0$ *such that*

$$\| \sum_k S(2k\pi i)^{-1} e^{2k\pi i \cdot} x_k \|_{W_p^n} \leq C \cdot \| \sum_k e^{2k\pi i \cdot} x_k \|_{W_p^m}; \tag{2.16}$$

for any finite sequence $\{x_k\} \subset E$

If E *is a Hilbert space ,* $p = 2$, *and* $0 \leq r \leq m$, *then (i) and (ii) are equivalent to*

(iii) For each $f \in W_p^r(J)$, *Equation* (1.1) *admits a 1-periodic mild solution, which belong to* $W_p^{n-m+r}(J)$*;*

(iv) For every $k \in \mathbb{Z}$, $(2k\pi i) \in \varrho(S)$ *and*

$$\sup_{k \in \mathbb{Z}} \|k^{n-m} S(2k\pi i)^{-1}\| < \infty \tag{2.17}$$

We will need the following lemma.

Lemma 2.8 *Let* $F_1 := W_p^m(J)$ *and* $F_2 := WP_p^n(J)$. *Then the following are equivalent:*

(1) For each function $f \in F_1$, (1.1) *admits a unique mild solution* u *in* F_2.

(2) There exists a dense subset D *in* F_1 *such that:*

(i) For each function $f \in D$, (1.1) *admits a unique mild solution* u *in* F_2*;*

(ii) There exists a constant $C > 0$ *such that*

$$\|u\|_{F_2} \leq C \|f\|_{F_1} \tag{2.18}$$

for all $f \in D$.

Proof. *(1)⇒ (2)*: Suppose for each function $f \in F_1$, (1.1) admits a unique mild solution u in F_2. Define the operator $G : F_1 \to F_2$ as follows: For each $f \in F_1$, Gf is the unique mild solution of (1.1) in F_2. By the assumptions, G is everywhere defined. We will show G is bounded by proving it is a closed operator.

To that end , let $\{f_l\}_{l>0}$ be a sequence in F_1 with $\lim_{l\to\infty} f_l = f$ in F_1 and $\lim_{l\to\infty} u_l = \lim_{m\to\infty} Gf_l = u$ in F_2. We will show $u = Gf$. Note that for each $t \in [0, 1]$ and $j = 0, 1, 2, ..., (n-1)$ we have $\lim_{l\to\infty} u_l^{(j)} = u^{(j)}(t)$, $\lim_{l\to\infty} \int_0^t u_l(s)ds = \int_0^t u(s)ds$ and $\lim_{l\to\infty} \int_0^t f_l(s)ds = \int_0^t f(s)ds$. Hence, if we denote

$$v_l(t) := \sum_{j=1}^{n-1} B_j u_l^{(j-1)}(t) + B_0 \int_0^t u_l(s)ds$$

then

$$\lim_{l\to\infty} v_l(t) = \sum_{j=1}^{n-1} B_j u^{(j-1)}(t) + B_0 \int_0^t u(s)ds.$$

Moreover,

$$\begin{aligned} Bv_l(t) &= B(\sum_{j=1}^{n-1} B_j u_l^{(i-1)}(t) + B_0 \int_0^t u_j(s)ds) \\ &= u_l^{(n-1)}(t) - u_l^{(n-1)}(0) - \int_0^t f_l(s)ds \\ &\to u^{(n-1)}(t) - u^{(n-1)}(0) - \int_0^t f(s)ds \end{aligned}$$

as $l \to \infty$. Since B is a closed operator, we have

$$\sum_{j=1}^{n-1} B_j u^{(j-1)}(t) + B_0 \int_0^t u(s)ds \in D(B)$$

and

$$B\Big(\sum_{j=1}^{n-1} B_j u^{(j-1)}(t) + B_0 \int_0^t u(s)ds\Big) = u^{(n-1)}(t) - u^{(n-1)}(0) - \int_0^t f(s)ds,$$

which means that u is a mild solution of (1.1) corresponding to f. So, G is a bounded operator and *(2)* is satisfied with $C = \|G\|$.

Conversely, suppose *(2)* is satisfied. Then, for any $f \in F_1$ there exists a sequence $\{f_l\} \subset D$ such that $f_l \to f$ in F_1 topology as $l \to \infty$. Let u_l be the mild solution to (1.1) in F_2 corresponding to f_l, then, by (2.18), $\lim_{l\to\infty} u_l = u$ for some $u \in F_2$ in F_2 topology and $\|u\|_{F_2} \leq C\|f\|_{F_1}$. With the same manner as the above part, we can show that u is a mild solution corresponding to f. The uniqueness of u can be easily clarified by using (2.18) and the lemma is proved.

Proof of Theorem 2.7. First we prove the following note: If x_0 and f_0 are two vectors in E with $S(2k\pi i)x_0 = f_0$, then $u(t) := e^{2k\pi it}x_0$ is a (classical) solution to (1.1) corresponding to $f(t) := e^{2k\pi it}f_0$. Indeed, $S(2k\pi i)x_0 = f_0$ means

$$\sum_{j=0}^{n-1}(2k\pi i)^j B_j x_0 \in D(B)$$

and

$$(2k\pi i)^n x_0 - B\sum_{j=0}^{n-1}(2k\pi i)^j B_j x_0 = f_0. \tag{2.19}$$

Multiply both side of (2.19) by $e^{2k\pi it}$ and note that $u^{(j)}(t) = (2k\pi i)^j e^{2k\pi it}x_0$, we have

$$u^{(n)}(t) - B\sum_{j=0}^{n-1} B_j u^{(j)}(t) = f(t),$$

which means u is a classical solution to (1.1).

(i) → (ii): We first show that $2k\pi i \in \varrho(S)$ for each $k \in \mathbb{Z}$. To that end, suppose x is any vector in E, $f(t) = e^{2k\pi it}x$ and let $u(t)$ be the unique mild 1-periodic solution to (1.1) corresponding to f, which is in $W_p^n(J)$. By Proposition 2.4 we have $S(2k\pi i)u_k = x$. Hence $S(2k\pi i)$ is surjective. On the other side, if $S(2k\pi i)$ is not invertible, i.e. there is a non-zero vector $x_0 \in E$ such that $S(2k\pi i)x_0 = 0$, then, by the above note, $u_1(t) :\equiv 0$ and $u_2(t) := e^{2k\pi it}x_0$ are two distinct 1-periodic classical, and hence mild solutions to the homogeneous equation $u^{(n)}(t) = \Sigma_{j=0}^{n-1} A_j u^{(j)}(t)$. This is contradicting to the uniqueness of u. So $S(2k\pi i)$ is invertible, i.e. $2k\pi i \in \varrho(S)$.

Let now $f(t) := \sum_k e^{2k\pi it}x_k$, where $\{x_k\}$ is any finite sequence in E. Then, by Proposition 2.4, $u(t) = \sum_k (S(2k\pi i)^{-1} e^{2k\pi it} x_k$ is the unique 1-periodic classical solution to (1.1) corresponding to f. Thus, (2.16) is obtained by inequality $\|u\|_{W_p^n} \leq \|G\| \cdot \|f\|_{W_p^m}$.

(ii) → *(i)*: Put

$$\mathcal{M} := \{f(t) = \sum_k e^{2k\pi it} x_k : \{x_k\} \text{ is a finite sequence in } E\}.$$

Observe that $\mathcal{M}$ is dense in $W_p^m(J)$. Moreover, if f is a function in $\mathcal{M}$, i.e., if $f(t) = \sum_k e^{2k\pi t} x_k$, then $u(t) = \sum_k (S(2k\pi i))^{-1} e^{2k\pi it} x_k$ is a 1-periodic classical solution of (1.1) corresponding to f and, by Corollary 2.6*(i)*, it is the unique one. From (2.16) it follows that $\|u\|_{W_p^n(J)} \leq C\|f\|_{W_p^m(J)}$ for all $f \in \mathcal{M}$. By Lemma 2.8, that implies *(i)*.

If E now is a Hilbert space, then $W_2^m(J)$ is a Hilbert space for any $0 \leq m \leq n$ with the norm

$$\|f\|^2_{W_2^m} = \sum_{j=0}^{m} \|f^{(j)}\|^2.$$

We first prove the equivalence $(ii) \Leftrightarrow (iv)$. Suppose *(ii)* holds. For any $k \in \mathbb{Z}$, take $f(t) := e^{2k\pi it} x$ and $u(t) = S(2k\pi i)^{-1} e^{2k\pi it} x$ be the corresponding solution to (1.1). We have

$$\|f\|^2_{W_2^m(J)} = \sum_{j=0}^{m} \|(2k\pi)^j x\|^2 \tag{2.20}$$

and

$$\|u\|^2_{W_2^n(J)} = \sum_{j=0}^{n} \|(2k\pi)^j S(2k\pi i)^{-1} x\|^2.$$

Using (2.16) we have

$$\sum_{j=0}^{n} \|(2k\pi)^j S(2k\pi i)^{-1} x\|^2 \leq C^2 \sum_{j=0}^{m} \|(2k\pi)^j x\|^2,$$

which implies

$$\|S(2k\pi i)^{-1} x\|^2 \leq C^2 \frac{\sum_{j=0}^{m} |2k\pi|^{2j}}{\sum_{j=0}^{n} |2k\pi|^{2j}} \cdot \|x\|^2 \tag{2.21}$$

for any $x \in E$ and any $k \in \mathbb{Z}$.

For a positive number λ and an integer m with $0 \le m \le n$ it is easy to show the inequality:

$$\frac{\sum_{j=0}^{m} \lambda^{2j}}{\sum_{j=0}^{n} \lambda^{2j}} \le \frac{1}{\lambda^{2(n-m)}}.$$

Thus, from (2.21) we have

$$\|S(2k\pi i)^{-1}x\| \le C\frac{1}{|2k\pi|^{n-m}} \cdot \|x\|$$

for all $k \in \mathbb{Z}$ and all $x \in E$, from which (2.17) follows.

Conversely, suppose (iv) holds, i.e., there is a constant $C > 0$ such that $\|S(2k\pi i)^{-1}\| \le C|k|^{m-n}$ for $k \in \mathbb{Z}$. Using that inequality we have

$$\begin{aligned}
\|\sum_k (S(2k\pi i)^{-1}e^{2k\pi i\cdot}x_k\|^2_{W_2^n(J)} &= \sum_{j=0}^{n}\Big(\sum_k (2k\pi)^{2j}\|S(2k\pi i)^{-1}x_k\|^2\Big) \\
&\le C\sum_{j=0}^{n}\Big(\sum_k (2k\pi)^{2j}k^{2m-2n}\|x_k\|^2\Big) \\
&\le C_1\sum_{j=0}^{n}\Big(\sum_k (2k\pi)^{2j+2m-2n}\|x_k\|^2\Big) \\
&= C_1\sum_k\Big(\sum_{j=0}^{n}(2k\pi)^{2j+2m-2n}\Big)\|x_k\|^2 \\
&\le C_1(n+1)\sum_k (2k\pi)^{2m}\|x_k\|^2 \\
&\le C_1(n+1)\sum_{j=0}^{m}\Big(\sum_k (2k\pi)^{2j}\|x_k\|^2\Big) \\
&= C_1(n+1)\|\sum_k e^{2k\pi i\cdot}x_k\|^2_{W_2^m(J)},
\end{aligned}$$

where $C_1 = C(2\pi)^{n-m}$. Thus, (2.16) holds and (ii) is satisfied.

Finally, observe that if E is a Hilbert space and $0 \le r \le m$, then with the same manner as in the proof $(ii) \Leftrightarrow (iv)$, we can show that (iv) is equivalent to (*ii'*) For each $k \in \mathbb{Z}$, $2k\pi i \in \varrho(S)$ and there exists a constant $C > 0$ such that

$$\| \sum_k (S(2k\pi i)^{-1} e^{2k\pi i \cdot} x_k)\|_{W_p^{n-m+r}} \leq C \cdot \| \sum_k e^{2k\pi i \cdot} x_k \|_{W_p^r} \tag{2.22}$$

for any finite sequence $\{x_k\} \subset E$.

On the other hand, (*ii'*) is equivalent to (*iii*) due to Lemma 2.8. Hence, (*iii*) is equivalent to (*iv*) and the theorem is completely proved. ♣

3 Some special cases

The $u^{(n)}(t) = Au(t) + f(t)$ case: We consider the higher order differential equation

$$u^{(n)}(t) = Au(t) + f(t), \quad 0 \leq t \leq 1, \tag{3.1}$$

where A is a linear, closed and densely defined operator with $\varrho(A) \neq \emptyset$. In this case, Condition F is satisfied, as we choose $B = (\lambda - A)$, where $\lambda \in \varrho(A)$. Indeed, $\overline{B^{-1}A} = \overline{(\lambda - A)^{-1}A} = \lambda(\lambda - A)^{-1} - \lambda I$ is bounded. Also, $S(\lambda) = (\lambda^n - A)$. Hence, applying Theorem 2.7, we have

Theorem 3.1 *The following statements are equivalent.*

(i) For each function $f \in W_p^m$, Equation (3.1) *admits a unique 1 periodic mild solution in W_p^n;*

(ii) For each $k \in \mathbb{Z}$, $(2k\pi i)^n \in \varrho(A)$ and there exists a constant $C > 0$ such that

$$\| \sum_k ((2k\pi i)^n - A)^{-1} e^{2k\pi i \cdot} x_k \|_{PW_p^{n-1}} \leq C \cdot \| \sum_k e^{2k\pi i \cdot} x_k \|_{W_p^m}; \tag{3.2}$$

for any finite sequence $\{x_k\} \subset E$

If E is a Hilbert space, $p = 2$ and $0 \leq r \leq m$, then (i) and (ii) are equivalent to *(iii) For each function* $f \in W_p^r$*, Equation* (3.1) *admits a unique 1-periodic mild solution in* W_p^{n-m+r}*;*

(iv) For every $k \in \mathbb{Z}$*,* $(2k\pi i)^n \in \varrho(A)$ *and*

$$\sup_{k\in\mathbb{Z}} \|k^{n-m}((2k\pi i)^n - A)^{-1}\| < \infty. \tag{3.3}$$

The Semigroup case: When $n = 1$ and A generates a C_0-semigroup $(T(t))_{t\geq 0}$, then the mild solution of the differential equation

$$u'(t) = Au(t) + f(t), \quad 0 \leq t \leq 1 \tag{3.4}$$

can be expressed by

$$u(t) = T(t)u(0) + \int_0^t T(t-s)f(s)ds. \tag{3.5}$$

We have the following result, in which the equivalence between *(i)* and *(v)* is the Gearhart's Theorem [6].

Theorem 3.2 *Let* A *generate a* C_0*-semigroup* $(T(t))_{t\geq 0}$*. Then the following statements are equivalent:*

(i) $1 \in \varrho(T(1))$*;*

(ii) For every function $f \in L_p(J)$*, Equation* (3.4) *admits a unique 1-periodic mild solution;*

(iii) For every function $f \in WP_p^1(J)$*, Equation* (3.4) *admits a unique mild solution in* $WP_p^1(J)$*;*

(iv) For every function $f \in WP_p^1(J)$*, Equation* (3.4) *admits a unique 1-periodic classical solution*

If E *is a Hilbert space, all the above statements are equivalent to*

(v) $\{2k\pi i : k \in \mathbb{Z}\} \subset \varrho(A)$ *and*

$$\sup_{k\in\mathbb{Z}} \|(2k\pi i - A)^{-1}\| < \infty.$$

Proof. The equivalence *(i)* $\Leftrightarrow$ *(ii)* was proved in [14]. The equivalence *(ii)* $\Leftrightarrow$ *(iv)* can be shown by using standard arguments and, if E is a Hilbert space, *(iii)* $\Leftrightarrow$ *(v)* follows from Theorem 3.1. The inclusion $(iv) \Rightarrow (iii)$ is obvious. So, it remains to show *(iii)* $\rightarrow$ *(iv)*.

To this end, let u be the unique mild solution of (3.4), which belong to $WP_p^1(J)$. Since $\int_0^t T(t-s)f(s)ds \in D(A)$ and $t \rightarrow \int_0^t T(t-s)f(s)ds$ is continuously differentiable for any $f \in W_p^1(J)$ (see e.g. [11]), we obtain that $T(\cdot)u(0) \in W_p^1(J)$. It follows that $T(t)u(0) \in D(A)$ for $t > 0$ (since $t \mapsto T(t)x$ is differentiable at t_0 if and only if $T(t_0)x \in D(A)$). Hence, $u(1)$, and thus, $x = u(1)$ belongs to $D(A)$. So u is a classical solution. The uniqueness of the 1-periodic classical solution is obvious. ♣

A cosine family case: We now consider the second order differential equation:

$$u''(t) = Au(t) + f(t) \quad 0 \leq t \leq 1, \tag{3.6}$$

where A is generator of a cosine family $(C(t))_{t\in\mathbb{R}}$ on E. Recall (see u.g. [1]) that in this case there exists a Banach space F such that $D(A) \hookrightarrow F \hookrightarrow E$ and such that the operator

$$\mathcal{A} := \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$$

with $D(\mathcal{A}) = D(A) \times F$ generates the C_0-semigroup $\mathcal{T}(t) := \begin{pmatrix} C(t) & S(t) \\ C'(t) & C(t) \end{pmatrix}$ on $F \times E$, where $S(t)$ is the associated sine family. Moreover, it is not hard to check that u is a mild solution of (3.6), which is continuously differentiable (a mild solution, which is in $WP_p^2(J)$, or a classical solution of (3.6), respectively), if and only if $\mathcal{U} = (u, u')^T$ is a mild solution (a mild solution, which is in $WP_p^1(J)$, or a classical solution, respectively) of the first order differential equation

$$\mathcal{U}'(t) = \mathcal{A}\mathcal{U}(t) + (0, f(t))^T, \quad 0 \leq t \leq 1 \tag{3.7}$$

in space $F \times E$. Using (3.5), we have the explicit form of u by

$$u(t) = C(t)u(0) + S(t)u'(0) + \int_0^t S(s-\tau)f(\tau)d\tau.$$

We have the following result, in which the equivalence between *(i)* and *(ii)* is proved in [17] and the equivalence among (*ii*), (*iii*), (*iv*) and (*v*) follows from the above observation and Theorem 3.2.

Theorem 3.3 *Let A generate a cosine family $(C(t))_{t\in\mathbb{R}}$ in E. Then the following statements are equivalent.*

(i) $1 \in \varrho(C(1))$;

(ii) For each function $f \in L_p(J)$, Equation (3.6) *has a unique 1-periodic mild solution, which is continuously differentiable;*

(iii) For each function $f \in WP_p^1(J)$, Equation (3.6) *admits a unique mild solution in $WP_p^2(J)$;*

(iv) For each function $f \in WP_p^1(J)$, Equation (3.6) *admits a unique 1-periodic classical solution;*

If E is a Hilbert space, all the above statements are equivalent to

(v) $\{-4k^2\pi^2 : k \in \mathbb{Z}\} \subset \varrho(A)$ *and*

$$\sup_{k\in\mathbb{Z}} \|k(4k^2\pi^2 + A)^{-1}\| < \infty.$$

A complete case: We consider the following differential equation:

$$\prod_{j=1}^{n}(\frac{d}{dt} - a_j A)u(t) = f(t), \quad 0 \le t \le 1, \tag{3.8}$$

where a_j, $j = 1, 2, ..., n$ are non-zero complex numbers and A is a linear and closed operator on E with $\varrho(A) \neq \emptyset$. We can re-write (3.8) as the following:

$$u^{(n)}(t) + \sum_{j=0}^{n-1} b_j A^{n-j} u^{(j)}(t) = f(t), \tag{3.9}$$

where b_j are certain, corresponding coefficients. So, in this case $A_j = -b_j A^{n-j}$ and they satisfy Condition F with $B = (\lambda - A)^n$. Moreover, by a short calculation, $S(\lambda)$ is

$$\begin{aligned} S(\lambda) &= \lambda^n + \sum_{j=0}^{n-1} \lambda^j b_j A^{n-j} \\ &= \prod_{j=1}^{n} (\lambda - a_j A) \end{aligned}$$

with $D(S(\lambda)) = D(A^n)$. It is not hard to show that $\lambda \in \varrho(S)$ if and only if $\lambda/a_j \in \varrho(A)$ and

$$S(\lambda)^{-1} = \prod_{j=1}^{n} (\lambda - a_{n-j+1}A)^{-1} = \prod_{j=1}^{n} (\lambda - a_j A)^{-1}.$$

Hence, from Theorem 2.7 we have:

Theorem 3.4 *Let A be a linear and closed operator on a Hilbert space E with $\varrho(A) \neq \emptyset$ and $0 \leq m \leq r \leq n$. Then, for each function $f \in W_2^m$, Equation* (3.8) *admits a unique 1-periodic mild solution in $W_2^r(J)$, if for all $k \in \mathbb{Z}$, the following conditions are satisfied:*

(i) $\frac{2k\pi i}{a_j} \in \varrho(A)$ for $j = 1, 2, ..., n$;

(ii) There exists a constant C such that $|k|^{(r-m)/n} \|(2k\pi i - a_j A)^{-1}\| < C$.

References

[1] ARENDT W., BATTY C.J.K., HIEBER M., NEUBRANDER F.: *Vector-valued Laplace Transforms and Cauchy Problems.* Birkhäuser Verlag, Basel-Boston-Berlin 2001.

[2] ARENDT, W., BATTY, C. J. K.: *Almost periodic solutions of first- and second-order Cauchy problems.* J. Differential Equations **137** (1997), no. 2, 363–383.

[3] CIORANESCU I., LIZAMA C.: *Spectral properties of cosine operator functions.* Aequationes Math. **36** (1988), no. 1, 80–98.

[4] Daleckii J., Krein M. G.: *Stability of solutions of differential equations on Banach spaces.* Amer. Math. Soc., Providence, RI, 1974.

[5] Eidelman Y. S., Tikhonov I. V.: *On periodic solutions of abstract differential equations* Abstr. Appl. Anal. **6** (2001), no. 8, 489–499.

[6] Gearhart L.: *Spectral theory for contraction semigroups on Hilbert space.* Trans. Amer. Math. Soc. **236** (1978), 385–394.

[7] Hale J. K.: *Ordinary differential equations.* Wiley-Interscience, New York, 1969.

[8] Hatvani, L. and Krisztin T.: *On the existence of periodic solutions for linear inhomogeneous and quasi-linear functional differential equations.* J. Differential Equations **97** (1992), 1–15.

[9] Haraux A.: *Nonlinear evolution equations* Lecture Notes in Math., vol. **841** Springer Verlag, Heidelberg 1981. (1992), 1–15.

[10] Latushkin Y, Montgomery-Smith S.: *Evolution semigroups and Liapunov Theorems in Banach spaces.* J. Func. Anal. **127** (1995), 173–197.

[11] Nagel, R., Sinestrari, E.: *Inhomogeneous Volterra integrodifferential equations for Hille–Yosida operators.* In: K.D. Bierstedt, A. Pietsch, W. M. Ruess, D. Vogt (eds.): Functional Analysis. Proc. Essen Conference, Marcel Dekker 1993, 51-70.

[12] S. Murakami, T. Naito and Nguyen Van Minh: *Evolution Semigroups and Sums of Commuting Operators: A New Approach to the Admissibility Theory of Function Spaces.* J. Differential Equations, 164, (2000), pp. 240-285.

[13] Nguyen L.: *Periodicity of Mild Solutions to Higher Order Differential Equations in Banach Spaces* Electronic J. Diff. Equ. **79** (2004), 1-12.

[14] Pruss J.: *On the spectrum of C_0-semigroup.* Trans. Amer. Math. Soc. **284**, 1984, 847–857 .

[15] Schüler E., Vu Q. P.: *The operator equation $AX - XB = C$, admissibility and asymptotic behavior of differential equations.* J. Differential Equations, **145** (1998), 394-419.

[16] Schüler E., Vu Q. P.: *The operator equation $AX - XD^2 = -\delta_0$ and second order differential equations in Banach spaces.* Semigroups of operators: theory and applications (Newport Beach, CA, 1998), 352363, Progr. Nonlinear Differential Equations Appl., 42, Birkhauser, Basel, 2000.

[17] Schüler E.: *On the spectrum of cosine functions.* J. Math. Anal. Appl. **229** (1999), 376–398.

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