On the Mild Solutions of Higher Order Differential Equations in Banach Spaces[§]

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Abstract

For the higher order differential equation $u^{(n)}(t) = Au(t) + f(t)$, $t \in \mathbb{R}$ (*) on a Banach space E, we give a new definition of mild solutions of (*). We then characterize the regular admissiblity of a translation invariant subspace $\mathcal{M}$ of $BUC(\mathbb{R}, E)$ with respect to (*) in terms of solvability of the operator equation $AX - XD^n = C$. As applications, periodicity and almost periodicity of mild solutions of (*) are also proved.

1 Introduction

The qualitative theory of mild solutions on the whole line of the differential equation of the type

$$u'(t) = Au(t) + f(t), \qquad t \in \mathbb{R}, \tag{1.1}$$

where A is a closed operator on a Banach space E, has been of increasing interest in the last decades. If A is a bounded operator on E, mild solutions of (1.1), which are the same as the classical solutions, are defined by

$$u(t) = e^{At}u(0) + \int_0^t e^{A(t-s)}f(s)ds, \quad t \in \mathbb{R}. \tag{1.2}$$

In their book [3], Daleckii and Krein made a systematic study on the asymptotic behavior of solutions of the form (1.2). For unbounded operator A,

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where the situation changes dramatically, the first question is, which solutions of (1.1) are considered as mild solutions. If A is the generator of a C_0-semigroup $T(t)$, $t \geq 0$, it is logical to define mild solutions of (1.1) by

$$u(t) = T(t-s)u(s) + \int_s^t T(t-\tau)f(\tau)d\tau, \quad t \geq s. \tag{1.3}$$

With this definition in hand, many authors investigated the qualitative behavior of (1.3) in different ways (see [8], [11], [14], [19], [20], and references therein). The second order differential equation, $u''(t) = Au(t) + f(t)$, where A is the generator of a cosine family $(C(t))$, and for which mild solutions are defined by

$$u(t) = C(t-s)u(s) + S(t-s)u'(s) + \int_s^t S(t-\tau)f(\tau)d\tau,$$

has been also studied in [9], [10] and [15].

Recently, Arendt and Batty [2], Schweiker [16], and Vu Quoc Phong and Schuler [21] studied the first and second order differential equation, in which A is not the generator of a C_0-semigroup or of a cosine family (respectively). Although their definitions of mild solutions are slightly different, they all showed that the existence and uniqueness of mild solutions, which belong to a subspace $\mathcal{M}$ of $BUC(\mathbb{R}, E)$, are closely related to the solvability of the operator equation of the form

$$AX - XD = -\delta_0,$$

where D is the differential operator in $\mathcal{M}$ and δ_0 is the Dirac operator defined by $\delta_0(f) := f(0)$.

Inspired by this rapid developement, in this paper, we consider the higher order differential equation

$$u^{(n)}(t) = Au(t) + f(t), \tag{1.4}$$

where A is a closed linear operator on E and f is a continuous function from $\mathbb{R}$ to E. First, we give a general definition of mild solutions to Equation (1.4). This definition is an extension of that introduced in [2], where $n = 1$

and $n = 2$, and A generally is not the generator of a C_0-semigroup (and of a cosine family, respectively). Several properies of mild solutions are then shown in Section 2.

In Section 3, we consider the conditions for the solvability of operator equation $AX - XB = C$, in particular, when $B = D^n$, where D is the differential operator on a function space, and $C = -\delta_0$.

Assume that $\mathcal{M}$ is a closed, translation-invariant subspace of $BUC(\mathbb{R}, E)$. $\mathcal{M}$ is said to be regularly admissible with respect to Equation (1.4), if for every $f \in \mathcal{M}$ Equation (1.4) has a unique mild solution $u \in \mathcal{M}$. In Section 4 we characterize the regular admissibility of $\mathcal{M}$ in terms of solvability of the operator equation. Namely, we show that the subspace $\mathcal{M}$ is regularly admissible if and only if the operator equation of the form

$$AX - XD^n = -\delta_0 \tag{1.5}$$

has a unique bounded solution. As applications, in Section 5 we show that if the admissible subspace $\mathcal{M}$ is the space of 1-periodic functions, then $\sup_{k \in \mathbb{Z}} \|k^m((2\pi ki)^n - A)^{-1}\| < \infty$ is a necessary condition, that each mild solution on $\mathcal{M}$ belongs to $C^{(m)}(\mathbb{R}, E)$, where $0 \leq m \leq n$. Finally, we prove that, under some classical condition, if $\sigma(A) \cap (i\mathbb{R})^n$ is countable, then each bounded mild solution of the higher order equation is almost periodic, provided f is almost periodic. This result, shown by a short proof, generalises [2, Theorem 4.5].

2 Mild Solutions of Higher Order Differential Equations

First let us fix some notations. By $C^{(n)}(\mathbb{R}, E)$ we denote the space of continuous functions with continuous derivatives $u', u'', \ldots u^{(n)}$, and by $BUC(\mathbb{R}, E)$ the space of bounded, uniformly continuous functions with values in E. The operator $I : C(\mathbb{R}, E) \to C(\mathbb{R}, E)$ is defined by $If(t) := \int_0^t f(s)ds$ and $I^n f := I(I^{n-1}f)$.

Definition 2.1 a) We say that $u : \mathbb{R} \to E$ is a classical solution of (1.4), if $u \in D(A)$, $u \in C^n(\mathbb{R}, E)$ and (1.4) is satisfied.

b) A continuous function $u(t) \in C(\mathbb{R}, E)$ is called a mild solution of (1.4), if $I^{(n)}u(t) \in D(A)$ for all $t \in \mathbb{R}$ and there exist n points $v_0, v_1, \ldots, v_{n-1}$ in E such that

$$u(t) = \sum_{i=0}^{n-1} \frac{t^i}{i!} v_i + A I^n u(t) + I^n f(t) \qquad (2.1)$$

for all $t \in \mathbb{R}$.

Remark. Using the standard argument, we can prove the following.

(i) If a mild solution u is m times differentiable, $0 \le m < n$, then v_i, $i = 0, 1, \ldots, m$, are the initial values, i.e. $u(0) = v_0$, $u'(0) = v_1$, …, and $u^{(m)}(0) = v_m$.

(ii) If $n = 1$ and A is the generator of a C_0 semigroup $T(t)$, then a continuous function $u : \mathbb{R} \to E$ is a mild solution of (1.4) if and only if it has the form

$$u(t) = T(t-s)u(s) + \int_s^t T(t-r)f(r)dr.$$

(iii) Similarly, if $n = 2$ and A a generator of a cosine family $(C(t))$ on E, any continuously differentiable function u on E of the form

$$u(t) = C(t-s)u(s) + S(t-s)u'(s) + \int_s^t S(t-\tau)f(\tau)d\tau,$$

where $(S(t))$ is the associated sine family, is a mild solution of (1.4).

(iv) If u is a bounded mild soution of (1.4) corresponding to a bounded inhomogenity f and $\varphi \in L^1(\mathbb{R}, E)$ then $u * \varphi$ is a mild solution solution of (1.4) corresponding to $f * \varphi$.

Directly from their definition, we can collect some properties of mild solutions of Equation (1.4).

Lemma 2.2 Let u be a mild solution of the higher order differentiable equation (1.4). If

(i) u is in $C^{(n)}(\mathbb{R}, E)$; or

(ii) $u(t) \in D(A)$ for all $t \in \mathbb{R}$ and $Au(\cdot) \in C(\mathbb{R}, E)$,

then u is a classical solution.

Proof. (i) Since u is a mild solution we have

$$AI^n u(t) = u(t) - \sum_{0}^{n-1} \frac{t^i}{i!} v_i - I^n f(t). \tag{2.2}$$

The right hand side of (2.2) is n-time differentiable, so is the left hand side. Hence,

$$\begin{aligned} \lim_{h \to 0} A \frac{1}{h} \int_t^{t+h} I^{n-1} u(s) ds &= \lim_{h \to 0} \frac{1}{h} \left(A \int_0^{t+h} I^{n-1} u(s) ds - A \int_0^t I^{n-1} u(s) ds \right) \\ &= \frac{d}{dt} (AI^n(t)) \end{aligned}$$

exists. Since $\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} I^{n-1} u(s) ds = I^{n-1} u(t)$ and A is closed, we obtain that $I^{n-1} u(t) \in D(A)$ and $\frac{d}{dt}(AI^n u(t)) = AI^{n-1} u(t)$. By taking the derivative on both sides of (2.2), we obtain

$$AI^{n-1}(t) = u'(t) - \sum_{0}^{n-2} \frac{t^i}{i!} v_{i+1} - I^{n-1} f(t)$$

for all $t \in \mathbb{R}$. Repeating this procedure $(n-1)$ times we obtain u is n times differentiable and $u^{(n)}(t) = Au(t) + f(t)$, i.e. u is a classical solution.

(ii) If $u(t) \in D(A)$ for all $t \in \mathbb{R}$ and $Au(\cdot) \in C(\mathbb{R}, E)$, then $AI^n u(t) = I^n A u(t)$. Taking the n^{th} derivative of the right hand side of

$$u(t) = \sum_{0}^{n-1} \frac{t^i}{i!} v_i + I^n A u(t) + I^n f(t),$$

we have u is n times continuously differentiable and $u^{(n)}(t) = Au(t) + f(t)$, i.e., u is a classical solution. ♣

In the following we consider the spectrum of mild solutions of (1.4). For a bounded function $u \in L^\infty(\mathbb{R}, E)$, the Carleman tranform $\hat{u}$ of u is defined by

$$\hat{u}(\lambda) = \begin{cases} \int_0^\infty e^{-\lambda t} u(t) dt, & Re(\lambda) > 0, \\ -\int_{-\infty}^0 e^{-\lambda t} u(t) dt, & Re(\lambda) < 0. \end{cases} \tag{2.3}$$

It is clear that $\hat{u}$ is holomorphic on $\mathbb{C} \setminus i\mathbb{R}$. A point $\mu \in \mathbb{R}$ is called a regular point if $\hat{u}$ has a holomorphic extension in a neighborhood of $i\mu$. The spectrum of u is defined as follows

$$sp(u) = \{\mu \in \mathbb{R} : \mu \text{ is not regular}\}$$

The following lemma, whose proof can be found in [6] and [12], will be needed later.

Lemma 2.3 Let f, g be in $BUC(\mathbb{R}, E)$ and $\varphi \in L^1(\mathbb{R}, E)$. Then

(i) $sp(f)$ is closed and $sp(f) = \emptyset$ if and only if $f = 0$.

(ii) $sp(f + g) \subset sp(f) \cup sp(g)$.

(iii) $sp(f * \varphi) \subset sp(f) \cap \operatorname{supp} F\varphi$, where $F\varphi$ is the Fourier fransform of φ.

The following lemma is the first result about the spectrum of mild solutions of Equation (1.4).

Lemma 2.4 Let f be a bounded continuous function and u be a bounded mild solution of (1.4). Then

$$sp(u) \subset \{\mu \in R : (i\mu)^n \in \sigma(A)\} \cup sp(f).$$

Proof. It is easy to see that $\hat{I}u(\lambda) = \frac{1}{\lambda}\hat{u}(\lambda)$, hence $\hat{I}^n u(\lambda) = \frac{1}{\lambda^n}\hat{u}(\lambda)$. Taking the Carleman transform on both sides of Equation (2.1) we have

$$\hat{u}(\lambda) = Q(\lambda) + \frac{1}{\lambda^n} A\hat{u}(\lambda) + \frac{1}{\lambda^n}\hat{f}(\lambda), \qquad (2.4)$$

where $Q(\lambda) = \int_0^\infty e^{-\lambda t}(\sum_{i=0}^{n-1} \frac{t^i}{i!} v_i) dt = \sum_{i=0}^{n-1} u_i/\lambda^i$. From Equation (2.4) we obtain

$$(\lambda^n - A)\hat{u}(\lambda) = \lambda^n Q(\lambda) + \hat{f}(\lambda)$$

for $\lambda \notin i\mathbb{R}$. Hence, for $\lambda^n \in \rho(A)$ we have

$$\hat{u}(\lambda) = (\lambda^n - A)^{-1}(\lambda^n Q(\lambda) + \hat{f}(\lambda)).$$

Note that $\lambda^n Q(\lambda)$ is a holomophic function in terms of λ. It implies that if $\mu \in \mathbb{R}$ is a regular point of f and $(i\mu)^n \in \rho(A)$, then $\hat{u}$ has holomorphic extension in a neighborhood of $i\mu$, i.e. μ is a regular point of u. Hence we have the inclusive relation. ♣

From Lemma 2.4, it directly follows.

Corollary 2.5 If u is a bounded mild solution of (1.4) corresponding to $f \equiv 0$, then $sp(u) \subset \{\mu \in \mathbb{R} : (i\mu)^n \in \sigma(A)\}$

Corollary 2.6 If $(i\mathbb{R})^n \cap \sigma(A) = \emptyset$, then (1.4) has at most one bounded mild solution.

3 The Equation $AX - XD^n = C$

Let A and B be closed, generally unbounded, linear operators on Banach spaces E and F, with dense domains $D(A)$ and $D(B)$, respectively, and let C be a bounded linear operator from E to F. A bounded operator $X : F \to E$ is called a solution of the operator equation

$$AX - XB = C \tag{3.1}$$

if for every $f \in D(B)$ we have $Xf \in D(A)$ and $AXf - XBf = Cf$. Equation (3.1) has been considered by many authors. It was first studied intensively for bounded operators by Daleckii and Krein [3], Rosenblum [13]. For unbounded case, (3.1) was studied in [1], [17], [19] and [20] when A and B are generators of C_0-semigroups, and in [14], [21] when A and B are closed operators. We cite here some main results, which will be used in the sequel.

Theorem 3.1 (i) ([17, Theorem 15]) Let A and B be generators of C_0-semigroups on E and F, one of which is analytic, such that $\sigma(A) \cap \sigma(B) = \emptyset$. Then for every bounded operator C, Equation (3.1) has a unique bounded solution.

(ii) ([14, Theorem 3.1]) Let A be a closed and B be a bounded operator such $\sigma(A) \cap \sigma(B) = \emptyset$. Then for every bounded operator C, Equation

(3.1) has a unique bounded solution X, which has the following integral form

$$X = \frac{1}{2\pi i}\int_{\Gamma} (\lambda - A)^{-1} C (\lambda - B)^{-1} d\lambda, \qquad (3.2)$$

where Γ is a closed Cauchy contour around $\sigma(B)$ and separated from $\sigma(A)$.

(iii) ([1, Theorem 2.1]) If Equation (3.1) has a unique bounded solution for every bounded operator C, then $\sigma(A) \cap \sigma(B) = \emptyset$.

We now consider the situation when $F = M$, a translation-invariant subspace of $BUC(\mathbb{R}, E)$ and $B = D^n_M$, the restriction of D^n to M, where $D := \frac{d}{dt}$ on $BUC(\mathbb{R}, E)$. It is well-known that $\sigma(D) = i\mathbb{R}$ and $\sigma(D^n) = (\sigma(D))^n$.

Let now $M_k := \{f \in M : sp(f) \subset [-ik, ik]\}$, $k \geq 1$. Then the following properies hold (See [4, 21]).

i) M_k are translation invariant subspaces,
ii) $M_k \subset M_{k+1}$ and
iii) D_{M_k} is bounded.

We first need the following Lemma, which was proved in [21].

Lemma 3.2 $\sigma(D_M) = \cup_{k=1}^{\infty} \sigma(D_{M_k})$.

From Lemma 3.2 we obtain the following

Lemma 3.3 $\sigma(D^n_M) = \cup_{k=1}^{\infty} \sigma(D^n_{M_k})$.

Proof. We show that

$$\sigma(D^n_M) \subset \cup_{k=1}^{\infty} \sigma(D^n_{M_k}). \qquad (3.3)$$

Note that $\sigma(D^n) = (i\mathbb{R})^n$, hence $\sigma(D^n_M) \subset (i\mathbb{R})^n$. Assume that $(i\lambda)^n \in \sigma(D^n_M)$, $\lambda \in \mathbb{R}$. Then there is a sequence of vectors $(f_k)_k \subset M$ such that $f_k \in D(D^n_M)$, $\|f_k\| = 1$ and

$$\lim_{k \to \infty} \|((i\lambda)^n - D^n_M) f_k\| = 0. \qquad (3.4)$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the n complex roots of the equation $x^n = (i\lambda)^n$. Then we have

$$((i\lambda)^n - D^n_M) f_k = \prod_{j=1}^{n} (\lambda_j - D_M) f_k.$$

We show that there is at least one λ_j belonging to the spectrum of D_M. Assume contrarily that all λ_j belong to $\rho(D_M)$, then

$$f_k = \prod_{j=1}^{n} (\lambda_j - D_M)^{-1}((i\lambda)^n - D_M^n)f_k \to 0 \text{ as } k \to \infty,$$

which is contradictory to $\|f_k\| = 1$. Hence there is a λ_j, which belongs to $\sigma(D_M)$. By Lemma 3.2, there is a number k such that $i\lambda_j \in \sigma(D_{M_k})$. Since D_{M_k} is bounded, $(i\lambda)^n = (i\lambda_j)^n \in \sigma(D_{M_k}^n)$ and hence, the inclusion (3.3) follows. Since the inverse of (3.3) is obvious, the lemma is proved. ♣

From Lemmas 3.2 and 3.3 it follows

Lemma 3.4 $\sigma(D_M^n) = \{\lambda^n : \lambda \in \sigma(D_M)\}$

We now return to the operator equation

$$AX - XD_M^n = \delta_0^M, \tag{3.5}$$

where δ_0^M is the restriction of the Dirac operator to M. Assume that

$$\sigma(A) \cap \{\lambda^n : \lambda \in \sigma(D_M)\} = \emptyset. \tag{3.6}$$

Then, by Lemma 3.4, it is equivalent to

$$\sigma(A) \cap \sigma(D_M^n) = \emptyset.$$

Therefore, for $k = 1, 2, \ldots$ we have

$$\sigma(A) \cap \sigma(D_{M_k}^n) = \emptyset.$$

By Theorem 3.1, the operator equation

$$AX - XD_{M_k}^n = \delta_0^{M_k}$$

has a unique bounded solution X_k, which is of the form

$$X_k = -\frac{1}{2\pi i}\int_{\Gamma_k} (\lambda - A)^{-1}\delta_0^{M_n}(\lambda - D_{M_k}^n)^{-1}d\lambda, \tag{3.7}$$

where Γ_k is a contour around $\sigma(D^n_{M_k})$ and separated from $\sigma(A)$. Moreover, the uniqueness of X_k implies

$$X_k|M_l = X_l \text{ for } l < k.$$

We state a result about the existence and uniqueness of bounded solutions of Equation (3.5), whose proof is similar to that of [21, Theorem 7] (for $n = 2$), and is omitted.

Theorem 3.5 Assume that condition (3.6) holds. Then the operator equation (3.5) has a unique bounded solution if and only if

$$\sup_{n\geq 1}\|X_k\| < \infty, \tag{3.8}$$

where X_k are defined by (3.7).

4 Admissible Subspaces

Let $\mathcal{M}$ be a closed translation-invariant subspace of $BUC(\mathbb{R},\mathbb{E})$, which is regularly admissible with respect to Equation (1.4). Define the linear operator G on $\mathcal{M}$ such that for each $f \in \mathcal{M}$, Gf is the unique mild solution of (1.4) in $\mathcal{M}$, we have the following.

Lemma 4.1 G is a linear, bounded operator on $\mathcal{M}$.

Proof. We define operator $\tilde{G} : \mathcal{M} \to \mathcal{M} \times \mathbb{E}^n$ by

$$\tilde{G}f := (u, v_0, v_1, \dots, v_{n-1}),$$

where u is the unique mild solution of (1.4) corresponding to f and $v_0, v_1, \dots, v_{n-1}$ are contained in the mild solution

$$u(t) = \sum_{0}^{n-1}\frac{t^i}{i!}v_i + AI^n u(t) + I^n f(t). \tag{4.1}$$

We will show that $\tilde{G}$ is closed. Let $(f_k)_{k\in\mathbb{N}} \subset \mathcal{M}$ with $\lim_k f_k = f$ and $\tilde{G}f_k = (u_k, v_{0,k}, \dots, v_{n-1,k})$ with $\lim_{k\to\infty} \tilde{G}f_k = (u, v_0, \dots, v_{n-1})$, i.e. $\lim_{k\to\infty} u_k = u$

and $\lim_{k\to\infty} v_{j,k} = v_k$ for $j = 0,1,\dots,n-1$. Then we have $\lim_{k\to\infty} I^n u_k(t) = I^n u(t)$ and, by Equation (4.1),

$$\begin{aligned} AI^n u_k(t) &= u_k(t) - \sum_{0}^{n-1} \frac{t^i}{i!} v_{i,k} - I^n f_k(t) \\ &\to u(t) - \sum_{0}^{n-1} \frac{t^i}{i!} v_k - I^n f(t) \text{ as } k \to \infty . \end{aligned}$$

Since A is closed we obtain that $I^n u(t) \in D(A)$ and

$$AI^n u(t) = u(t) - \sum_{0}^{n-1} \frac{t^i}{i!} v_i - I^n f(t).$$

That means $\tilde{G} f = (u, v_0, v_1, \dots, v_{n-1})$. Hence, $\tilde{G}$ is closed and thus bounded. Since $G = \tilde{G} \circ P$, where $P : M \times E^n \to M$ is the projection on the first coordinate and thus a bounded operator, we obtain that G is bounded. ♣

Operator G is called the solution operator of Equation (1.4). G is commuting with the translation, and hence, commuting with the differential operator, as the following lemma shows.

Lemma 4.2 Let A be a closed operator on E with non-empty resolvent set and M be an admissible subspace of $BUC(\mathbb{R}, E)$. Then the following holds.

i) $S_h \cdot G = G \cdot S_h$, where S_h is the translation operator on M.

ii) $D_M \cdot G = G \cdot D_M$

Proof. i) Let $u = Gf$ be the unique mild solution of the higher order differential equation (1.4). If u is a classical solution, then $(Gf)^{(n)}(t+h) = A(Gf)(t+h) + f(t+h)$, and hence, $S_h \cdot Gf = G \cdot S_h f$. For the case that u is not a classical solution, let $\lambda \in \rho(A)$. Since

$$R(\lambda, A)u(t) = \sum_{0}^{n-1} \frac{t^i}{i!} R(\lambda, A)u_i + AI^n R(\lambda, A)u(t) + I^n R(\lambda, A)f(t),$$

it is easy to see that $\tilde{u}(t) = R(\lambda, A)u(t)$ is the unique solution of (1.4) corresponding to $\tilde{f} = R(\lambda, A)f$. But $\tilde{u}(t) \in D(A)$ for all $t \in \mathbb{R}$. Hence, by

Lemma 2.2(ii), $\tilde{u}$ is a classical solution. From the above result for a classical solution and the fact that S_h and $R(\lambda,A)$ commute, we have

$$\begin{aligned} R(\lambda,A)S_hGf &= S_hR(\lambda,A)Gf = S_hGR(\lambda,A)f \\ &= GS_hR(\lambda,A)f = GR(\lambda,A)S_hf = R(\lambda,A)GS_hf, \end{aligned}$$

from which it follows $S_hGf = GS_hf$ for all $f \in M$. Part ii) is a direct consequence of i), and the lemma is proved ♣

Corollary 4.3 Assume that A is a closed operator with non-empty resolvent set. Let M be a regularly admissible subspace of $BUC(\mathbb{R},E)$ and u be the unique mild solution corresponding to f in M. If $f \in C^n(\mathbb{R},E)$ such that $f', f'', \ldots, f^{(n)}$ belong to M, then u is a classical solution.

In what follows, we assume that M satisfies the following additional assumption:

$$\text{For all } C \in L(M,E) \text{ and } f \in M, \tag{4.2}$$

$$\text{the function } \Phi(t) = CS(t)f \text{ belongs to } M.$$

The regular admissibility of a space is closely related to the solvability of operator equation (3.1). This relation was shown in [20], when $n = 1$, and in [16] and [21], when $n = 2$. The following theorem is a generalization of those results.

Theorem 4.4 Let A be a closed operator on E with non-empty resolvent set and M be a translation invariant subspace in $BUC(\mathbb{R},E)$, which satisfies the assumption (4.2). Then the following are equivalent.

(i) M is a regularly admissible.

(ii) The operator equation

$$AX - XD_M^{(n)} = -\delta_0 \tag{4.3}$$

has a unique solution.

(iii) For every bounded operator $C : M \to E$, the operator equation

$$AX - XD_M^{(n)} = C \tag{4.4}$$

has a unique solution.

Proof (i) $\Rightarrow$ (ii). Let $G : M \to M$ be the bounded operator defined by $Gf = u$ where u is the unique mild solution in M. We define the operator $X : M \Rightarrow E$ by

$$Xf := (Gf)(0).$$

Then X is a bounded operator. Now let $f \in D_M^n$. By Lemma 4.3, $u = Gf$ is a classical solution of (1.4), i.e.,

$$(Gf)^{(n)}(t) = A(Gf)(t) + f(t). \tag{4.5}$$

Note that, by Lemma 4.2, $(Gf)^{(n)} = Gf^{(n)}$. Taking $t = 0$ from (4.5) and using this fact, we have $AXf - XD^n f = -\delta_0 f$ for $f \in D_M^n$, i.e. X is a bounded solution of (4.3).

To show the uniqueness, we assume that X_0 is a solution of Equation (4.3). Then for every $f \in D_M^n$, the function $u \in M$, defined by $u(t) = X_0 S(t)f$, is a classical solution of Equation (1.4). Indeed,

$$u^{(n)}(t) = X_0 D^n S(t)f = (AX_0 + \delta_0)S(t)f = Au(t) + f(t)$$

for all $t \in \mathbb{R}$. We will show that $u(t) = X_0 S(t)f$ is a mild solution of (1.4) for every $f \in M$. To this end, let $f \in M$ and $(f_k)_{k \in \mathbb{N}} \in D(D_M^n)$ with $\lim_k f_k = f$. Then $Gf = \lim_k Gf_k = \lim_k X_0 S(\cdot)f_k = X_0 S(\cdot)f$. Hence, $Gf = X_0 S(\cdot)f$, i.e., $u = X_0 S(\cdot)f$ is a mild solution of (1.4).

Assume now that X_1 and X_2 are two solutions of (4.3). Then, for every $f \in M$, $u = (X_1 - X_2)S(\cdot)f$ is a mild solution of the higher order equation $u^{(n)}(t) = Au(t)$. By the uniqueness of the mild solution we have $u \equiv 0$, which implies $X_1 = X_2$.

(ii) $\Rightarrow$ (iii) Let X be the unique solution of (4.3). Define the bounded operator $Y : M \to E$ by $Yf := X\tilde{f}$, where $\tilde{f}(t) = -CS(t)f$. Let $f \in D(D_M^n)$, then $(D_M^n f)\tilde{}(t) = -CS(t)D_M^n f = D_M^n \tilde{f}(t)$. Hence we have

$$AYf = AX\tilde{f} = XD_M^n \tilde{f} + \delta_0 \tilde{f} = X(D_M^n f)\tilde{} + Cf = YD_M^n f + Cf,$$

i.e. Y is a bounded solution of (4.4).

The uniqueness of the solution of operator equation $AX - XD^n_M = C$ follows directly from the uniqueness of the solution of $AX - XD^n_M = -\delta_0$.

(iii) $\Rightarrow$ (i) We have shown above that, if X is a bounded solution of (4.3), then $u(t) := XS(t)f$ is a mild solution of the higher order equation (1.4). It remains to show that this solution is unique. In order to do it, assume that u is a mild solution of the homogeneous equation $u^{(n)}(t) = Au(t), t \in \mathbb{R}$. By Corollary 2.5, $(isp(u))^n \subset \sigma(A)$. On the other hand, since $u \in M$, $isp(u) \subset \sigma(D_M)$, which implies $(isp(u))^n \subset \sigma(D^n_M)$. By Theorem 3.1(iii), it follows from (iii) that $\sigma(A) \cap \sigma(D^n_M) = \emptyset$. Hence, $sp(u) = \emptyset$, so $u \equiv 0$ and the theorem is proved. ♣

5 Applications

In this section, we will apply the results in Section 4 to the space of periodic and of almost periodic functions. Let $P(\omega)$ be the space of periodic functions from $\mathbb{R}$ to E with the period ω. For the sake of simplicity, we assume the period $\omega = 1$. We begin with the case, in which $n = 2$ and A is the generator of a cosine family $(C(t))$. It is well-known that

(1) A is the generator of an analytic C_0-semigroup given by

$$e^{Az}x = \frac{1}{\sqrt{(\pi z)}} \int_0^\infty e^{-\frac{t^2}{4z}} C(t)x dt, \quad Re(z) > 0;$$

(2) D^2 is the generator of a cosine family given by

$$C(t) = \frac{1}{2}(S(t) + S(-t)),$$

and hence, is the generator of an (analytic) C_0-semigroup in $P(1)$.

By Theorem 3.1(i) and Theorem 4.4, $P(1)$ is regularly admissible if and only if $\sigma(A) \cap \sigma(D^2_{P(1)}) = \emptyset$. On the other hand, $\sigma(D^2_{P(1)}) = \{(2k\pi i)^2 : k \in \mathbb{Z}\} = \{-k^2\pi^2 : k \in \mathbb{Z}\}$. Hence, we have

Theorem 5.1 Let A be the generator of a strongly continuous cosine family. Then $P(1)$ is regularly admissible wrt. $u''(t) = Au(t) + f(t)$ if and only if $\{-4k^2\pi^2 : k \in \mathbb{Z}\} \subset \rho(A)$.

In general, however, the condition of the form $\sigma(A) \cap \sigma(D^n_M) = \emptyset$ does not imply the regular admissibility of space M. At least the operator A must satisfy some conditions, as the following theorem shows.

Theorem 5.2 Let A be a closed operator on a Banach space E with non-empty resolvent set and suppose $P(1)$ is regularly admissible with respect to the equation

$$u^{(n)}(t) = Au(t) + f(t), \quad t \in \mathbb{R}. \tag{5.1}$$

Then

(1) $(2\pi ki)^n \in \rho(A)$ and $\sup_{k\in\mathbb{Z}} \|((2\pi ki)^n - A)^{-1}\| < \infty$,

(2) if each mild solution on $P(1)$ belongs to $C^{(m)}(\mathbb{R}, E)$, $0 \le m \le n$, then $(2\pi ki)^n \in \rho(A)$ and $\sup_{k\in\mathbb{Z}} \|k^m((2\pi ki)^n - A)^{-1}\| < \infty$.

Proof. By assumption, $P(1)$ is a regularly admissible function space, so, by Theorem 4.4, the Equation $AX - XD^n_{P(1)} = C$ has a unique solution for every bounded operator C. Hence, by Theorem 3.1(iii), $\sigma(A) \cap \sigma(D^n_{P(1)}) = \emptyset$. On the other hand, it is not hard to see that $\sigma(D^n_{P(1)}) = \{(2k\pi i)^n : k \in \mathbb{Z}\}$. It follows that $\sigma(A) \cap \{(2k\pi i)^n : k \in \mathbb{Z}\} = \emptyset$, or in other words, $\{(2k\pi i)^n : k \in \mathbb{Z}\} \subset \rho(A)$.

To prove (1), let $G : P(1) \to P(1)$ be the solution operator and take $f(t) = e^{2k\pi it}x_0$, $x_0 \in E$, as a 1-periodic function. It is not too hard to check that $Gf(t) = e^{2k\pi it} \cdot ((2k\pi i)^n - A)^{-1}x_0$ is the (unique) mild solution of (5.1). Hence,

$$\|((2k\pi i)^n - A)^{-1}x_0\| = \|Gf\| \le \|G\| \cdot \|f\| = \|G\| \cdot \|x_0\|$$

for all $x_0 \in E$ and $k \in \mathbb{Z}$. Hence $\sup_{k\in\mathbb{Z}} \|((2k\pi i)^n - A)^{-1}\| \le \|G\| < \infty$.

To prove (2) observe that, since each mild solution on $P(1)$ belongs to

$C^{(m)}(\mathbb{R},E)$, the composite operator $D^m_{P(1)}G$ is everywhere defined and closed. Hence, it is a bounded operator. Thus,

$$\|D^m_{P(1)}Gf\| = \|(2k\pi)^m((2k\pi i)^n - A)^{-1}x_0\| \le \|D^m_{P(1)}G\|\cdot\|f\| = \|D^m_{P(1)}G\|\cdot\|x_0\|$$

for all $x_0 \in E$ and $k \in \mathbb{Z}$. Hence $\sup_{k\in\mathbb{Z}}\|k^m((2k\pi i)^n - A)^{-1}\| \le C\|D^m_{P(1)}G\|$ for a certain constant C, and that completes the proof. ♣

The converse of Theorem 5.2 does generally not hold (see [5] for a counter example). However, we have the affirmative answer in certain special cases. If E is a Hilbert space, $n = 1$ and A is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ we have the following theorem, whose proof of (b) ⇒ (a) can be found in [11].

Theorem 5.3 Let A be the generator of a C_0-semigroup on a Hilbert space E. Then the following are equivalent.

(a) For each 1-periodic function f, equation

$$u'(t) = Au(t) + f(t)$$

has a unique 1-periodic mild solution.

(b) $\{2\pi ki : k \in \mathbb{Z}\} \subset \rho(A)$ and $\sup_{k\in\mathbb{Z}}\|(2\pi ki - A)^{-1}\| < \infty$.

Also, if $n = 2$, $m = 1$ and A is the generator of a cosine family $(C(t))$ on a Hilbert space, we have a positive answer. Namely, we have the following theorem, whose proof of the converse part (b) ⇒ (a) can be found in [10].

Theorem 5.4 If A is the generator of a cosine family on a Hilbert space E, then the following are equivalent.

(a) For each 1-periodic function f, equation

$$u''(t) = Au(t) + f(t) \tag{5.2}$$

has a unique 1-periodic mild solution, which belongs to $C^1(\mathbb{R},E)$.

(b) $\{-4\pi^2k^2 : k \in \mathbb{Z}\} \subset \rho(A)$ and $\sup_{k\in\mathbb{Z}}\|k(4\pi^2k^2 + A)^{-1}\| < \infty$.

We now apply the results in Chapter 4 to $AP(\mathbb{R},E)$, the space of almost periodic functions from $\mathbb{R}$ to E. As preparation, we recall some basic concepts and results about almost periodic functions. (For more details, readers are refered to [2, 7]). A point $\lambda \in \mathbb{R}$ is called a point of almost periodicity of the function u, if there is a neighborhood U of λ such that for every $\varphi \in L^1(\mathbb{R})$ with $\operatorname{supp} F\varphi \subset U$, where $F\varphi$ is the Fourier transform of φ, the function $\varphi * u$ is almost periodic. The complement in $\mathbb{R}$ of the set of points of almost periodicity of u is called the almost periodic spectrum of f and is denoted by $sp_{AP}(u)$.

We say that $u \in BUC(\mathbb{R},E)$ is totally ergodic if $\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{-i\nu s}u(s)ds$ exists for all $\nu \in \mathbb{R}$. The following theorem can be found in [7] (part (a) and (b)) and [14] (part (c)).

Theorem 5.5 Let $u \in BUC(\mathbb{R},E)$ such that $sp_{AP}(u)$ is countable. Assume that

(a) $E \not\supset c_0$; or

(b) The range of $u(t)$ is weakly relatively compact; or

(c) u is totally ergodic.

Then u is almost periodic.

We now return to our higher order equation. Let Γ be a compact set in $\mathbb{R}$ and $M = X(\Gamma)$ be the subspace of $BUC(\mathbb{R},E)$ consisting of all functions f with $sp(f) \subset \Gamma$. It is easy to see that M satisfies condition (4.2). Moreover, D_M is bounded, $\sigma(D_M) = i\Gamma$ and thus, $\sigma(D^n_M) = (i\Gamma)^n$. Assume now $\sigma(A) \cap (i\Gamma)^n = \emptyset$, then, by Theorem 3.1(ii), the equation $AX - XD^n_M = -\delta_0$ has a unique solution. By Theorem 4.4, M is regularly admissible and for any almost periodic function f, the mild solution $u(t) = XS(t)f$ is also almost periodic. Using these facts we have the following

Theorem 5.6 For the equation

$$u^{(n)}(t) = Au(t) + f(t), \quad t \in \mathbb{R}, \tag{5.3}$$

we assume that f is almost periodic and $\sigma(A) \cap (i\mathbb{R})^n$ is countable. Let $u \in BUC(\mathbb{R}, E)$ be a mild solution of Equation (5.3). Then u is almost periodic if one of the following conditions is satisfied.

(a) $E \not\supset c_0$; or

(b) The range of $u(t)$ is weakly relatively compact; or

(c) u is totally ergodic.

Proof In view of Theorem 5.5, we have only to show that $sp_{AP}(u)$ is countable. Since $\sigma(A) \cap (i\mathbb{R})^n$ is countable, it suffices to prove that $(isp_{AP}(u))^n \subset \sigma(A)$.

Let λ be any point in $\mathbb{R}$ such that $(i\lambda)^n \in \rho(A)$, we will show that $\lambda \notin sp_{AP}(u)$. Since $\rho(A)$ is an open set, there exists $\varepsilon > 0$ such that $(i\Gamma)^n \subset \rho(A)$, where $\Gamma = [\lambda - \varepsilon, \lambda + \varepsilon]$. Since Γ is compact and $\sigma(A) \cap (i\Gamma)^n = \emptyset$, $X(\Gamma)$ is regularly admissible with respect to Equation (5.3).

Let φ be a function in $L^1(\mathbb{R}, E)$ with $\mathrm{supp} F\varphi \subset \Gamma$, and define $\tilde{u} := u * \varphi$ and $\tilde{f} := f * \varphi$. Then $\tilde{u}$ and $\tilde{f}$ are in $X(\Gamma)$ (Lemma 2.3(iii)) and $\tilde{f}$ is an almost periodic function. Moreover, $\tilde{u}$ is the unique mild solution of (5.3) corresponding to $\tilde{f}$ in $X(\Gamma)$ (Remark in Section 1). By the reasoning preceding this theorem, $\tilde{u}$ is also almost periodic. So, λ is a point of almost periodicity of u, i.e. $\lambda \notin sp_{AP}(u)$, and the theorem is proved. ♣

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