Operator Equation $AX - BXD = C$ and Degenerate Differential Equations in Banach Spaces

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Abstract

For the degenerate differential equation $\frac{d}{dt}Bu(t) = Au(t)+f(t), t \in \mathbb{R}$ (*) on a Banach space E, we characterize the regular admissibility of a subspace of $BUC(\mathbb{R}, E)$ in terms of solvability of the operator equation of the type $AX - XBD = C$. As applications, we prove the existence of Λ–class mild solutions of (*), in particular, of almost periodic, asymptotic almost periodic, etc. solutions.

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1 Introduction

The qualitative theory of mild solutions on the line of the differential equation of the type

$$u'(t) = Au(t) + f(t), \qquad t \in \mathbb{R}, \tag{1.1}$$

where A is a closed operator on a Banach space E, has been of increasing interest in the last decades. If A is a bounded operator on E, mild solutions of (1.1), which are the same as the classical solutions, are defined by

$$u(t) = e^{At}u(0) + \int_0^t e^{A(t-s)} f(s)ds, \quad t \in \mathbb{R}. \tag{1.2}$$

In their book [4], Daleckii and Krein made a systematic study on the asymptotic behavior of solutions of the form (1.2). For unbounded operator A, where the situation changes dramatically, the first question is, which solutions of (1.1) are considered as *mild solutions.* If A is the generator of a C_0-semigroup $T(t)$, $t \geq 0$, mild solutions of (1.1) are defined by

$$u(t) = T(t-s)u(s) + \int_s^t T(t-\tau) f(\tau)d\tau, \quad t \geq s. \tag{1.3}$$

With this definition in hand, many authors investigated the qualitative behavior of (1.3) in different ways (see [16], [18], [23], [24], [26], and references therein).

In this paper, we are concerned with the degenerate differential equation

$$\frac{d}{dt}Bu(t) = Au(t) + f(t), \tag{1.4}$$

where A and B are closed, densely defined operators on E and f is a continuous function from $\mathbb{R}$ to E. Equation (1.4) with initial condition $Bu(0) = u_0$, which is called the degenerate Cauchy problem (B may have no bounded inverse), or also the equation of *Sobolev type*, has been studied by many authors (see e.g. [6], [7], [8], [19], [20], [25], [27] and references therein). In this paper, we consider (1.4) on the line $\mathbb{R}$. First, we give a general definition of mild solutions to (1.4). This definition is an extension of that introduced in [1], where $B = Id$ and A generally is not the generator of a C_0-semigroup. Several properties of mild solutions are then shown in Section 2.

Assume that $\mathcal{M}$ is a closed, translation-invariant subspace of $BUC(\mathbb{R}, E)$. $\mathcal{M}$ is said to be *regularly admissible* with respect to Equation (1.4), if for every $f \in \mathcal{M}$ Equation (1.4) has a unique mild solution $u \in \mathcal{M}$, and the mild solutions are continuously dependent on the inhomogeneity. Closely related to the regular admissibility of $\mathcal{M}$ to Equation (1.4) is the solvability of the operator equation

$$AX - BX\mathcal{D} = -\delta_0. \tag{1.5}$$

For this purpose, in Section 3, we consider the operator equation of type

$$AX - BXD = C, \tag{1.6}$$

where A and B are linear (generally unbounded) operators on E, D is an operator on F and C is a bounded operator from F to E, being two Banach spaces. In order to study Equation (1.6), we investigate $\sigma(A, B)$, the *spectrum* of the operator pencil $\lambda B - A$. Moreover, we define the operators $\tau_{A,B,D}$ and τ_B on $L(E, F)$ by

$$\tau_{A,B,D}X := AX - BXD$$

and

$$\tau_B X := BX,$$

and show that

$$\sigma(A, B) - \sigma(D) \subset \sigma(\tau_{A,B,D}, \tau_B). \tag{1.7}$$

In particular, from (1.7) we obtain that, if Equation (1.6) has a unique bounded solution for every bounded operator C, i.e. $0 \notin \sigma(\tau_{A,B,D}, \tau_B)$, then $\sigma(A, B) \cap \sigma(D) = \emptyset$. The proofs in this section are based on the technique introduced in [2].

In Section 4 we characterize the regular admissibility of $\mathcal{M}$ in terms of solvability of the operator equation. Namely, we show that a subspace $\mathcal{M}$ is regularly admissible w.r.t. (1.4) if and only if, for every bounded operator C, Equation(1.6) has a unique bounded solution. As applications, we derive several results on particular subspaces. Among those results we obtain that, if the admissible subspace $\mathcal{M}$ is the space of continuous, periodic functions with period ω, then $\sup\limits_{k\in\mathbb{Z}} \|(\frac{2\pi ki}{\omega}B - A)^{-1}\| < \infty$ is a necessary condition for the regular admissibility of $\mathcal{M}$. Related results are in [26], where A generates a C_0-semigroup.

In Section 5, we prove that under certain classical conditions, each bounded mild solution of Equation (1.4) is of Λ-class if f is of Λ-class. This result, shown by a short proof, generalizes a result of Batty and Arendt [1, Theorem 4.5].

2 Mild Solutions of the Degenerate Equation

Let us first introduce the concept of the spectrum of an operator pencil. Suppose A and B are two linear, closed operators on E. With $Dom(A)$ we denote the domain of operator A. The operator pencil $\lambda B - A$ with $Dom(\lambda B - A) := Dom(A) \cap Dom(B)$ is defined by

$$(\lambda B - A)x := \lambda Bx - Ax.$$

By the resolvent set $\varrho(A, B)$, we denote the set of all $\lambda \in \mathbb{C}$, such that $\lambda B - A$ is bijective and $(\lambda B - A)^{-1}$ is bounded in E. The spectrum of $(\lambda B - A)$ is defined by $\sigma(A, B) := \mathbb{C} \setminus \varrho(A, B)$. The bounded operator $R(\lambda) := B(\lambda B - A)^{-1}$ is called the *resolvent operator*. Moreover, the point spectrum, the approximate spectrum, and the residual spectrum are defined as the following.

$$\sigma_p(A, B) := \{\lambda \in \mathbb{C} : \exists x \neq 0,\ Ax = \lambda Bx\}.$$

$$\begin{aligned} \sigma_{ap}(A, B) : &= \{\lambda \in \mathbb{C} : \text{There is a sequence } (x_n)_n \text{ such that } \|Bx_n\| = 1 \\ & \quad \text{and } (\lambda B - A)x_n \to 0\}. \end{aligned}$$

$$\sigma_r(A, B) := \{\lambda \in \mathbb{C} : \text{Range}(\lambda B - A) \text{ is not dense}\}.$$

It is not hard to see that $\sigma_p(A, B)$, $\sigma_{ap}(A, B)$ and $\sigma_r(A, B)$ are contained in $\sigma(A, B)$. Moreover, we have

Lemma 2.1 *Let A and B be closed operators on E. Then*

(i) $\varrho(A, B)$ is an open set in $\mathbb{C}$;

(ii) The approximate spectrum $\sigma_{ap}(A, B)$ contains the boundary $\partial\sigma(A, B)$ of $\sigma(A, B)$;

(iii) $\lambda \to R(\lambda)$ *and* $\lambda \to (\lambda B - A)^{-1}$ *are holomorphic functions on* $\varrho(A, B)$. *Moreover, the following equality holds*

$$R(\lambda) - R(\mu) = (\mu - \lambda) R(\mu) R(\lambda).$$

Proof. (i) Let $\lambda \in \varrho(A, B)$ and $|\mu - \lambda| \square\geq 1/\|B(\lambda B - A)^{-1}\|$. Then

$$(\mu B - A) = (\lambda B - A) - (\lambda - \mu)B = [I - (\lambda - \mu)B(\lambda B - A)^{-1}](\lambda B - A)$$

is invertible. Moreover,

$$(\mu B - A)^{-1} = (\lambda B - A)^{-1}[I - (\lambda - \mu)B(\lambda B - A)^{-1}]^{-1} \tag{2.8}$$

is a bounded operator. Hence, $\mu \in \varrho(A, B)$ and therefore, $\varrho(A, B)$ is open.

(ii) Let $\lambda_0 \in \partial\sigma(A, B)$. Then we can find a sequence $(\lambda_n)_{n \in \mathbb{N}} \subset \varrho(A, B)$ such that $\lambda_n \to \lambda_0$. From the proof of part (i), we obtain

$$\|B(\lambda_n B - A)^{-1}\| \geq 1/|\lambda_0 - \lambda_n| \to \infty$$

as $n \to \infty$. By the uniform boundedness theorem, there is a vector $x \in X$ such that $\|x\| = 1$ and $\|B(\lambda_n B - A)^{-1}x\| \to \infty$ as $n \to \infty$. Define

$$x_n := (\lambda_n B - A)^{-1}x / \|B(\lambda_n B - A)^{-1}x\|,$$

then $\|Bx_n\| = 1$ and

$$(\lambda_0 B - A)x_n = (\lambda_n B - A)x_n + (\lambda_0 - \lambda_n)Bx_n \to 0$$

as $n \to \infty$. Hence, $\lambda_0 \in \sigma_{ap}(A, B)$.

(iii) From (2.8) it follows

$$\begin{aligned} R(\mu) - R(\lambda) &= B(\lambda B - A)^{-1}[(I - (\lambda - \mu)B(\lambda B - A)^{-1}) - I] \\ &= B(\lambda B - A)^{-1} \sum_{k=1}^{\infty} (\lambda - \mu)^k [B(\lambda B - A)^{-1}]^k \\ &\to 0 \text{ as } \mu \to \lambda. \end{aligned}$$

Hence, $R(\lambda)$ is continuous. Furthermore,

$$\begin{aligned} R(\lambda) - R(\mu) &= B[I - (\mu B - A)^{-1}(\lambda B - A)](\lambda B - A)^{-1} \\ &= B(\mu B - A)^{-1}[\mu B - A - (\lambda B - A)](\lambda B - A)^{-1} \\ &= (\mu - \lambda)[B(\mu B - A)^{-1}][B(\lambda B - A)^{-1}] \\ &= (\mu - \lambda)R(\mu)R(\lambda), \end{aligned}$$

which implies that $R(\lambda)$ is holomorphic. Similarly, the continuity of $\lambda \to (\lambda B - A)^{-1}$ comes from

$$\begin{aligned} (\lambda B - A)^{-1} - (\mu B - A)^{-1} &= (\lambda B - A)^{-1}[(I - (\lambda - \mu)B(\lambda B - A)^{-1}) - I] \\ &= (\lambda B - A)^{-1}\sum_{k=1}^{\infty}(\lambda - \mu)^k[B(\lambda B - A)^{-1}]^k \\ &\to 0 \text{ as } \mu \to \lambda. \end{aligned}$$

Finally, from the identity

$$(\lambda B - A)^{-1} - (\mu B - A)^{-1} = (\mu - \lambda)(\lambda B - A)^{-1}B(\mu B - A)^{-1}$$

it follows that $\lambda \to (\lambda B - A)^{-1}$ is a holomorphic function on $\varrho(A, B)$. ♣

We now introduce *classical* and *mild* solutions of the degenerate equation

$$\frac{d}{dt}Bu(t) = Au(t) + f(t). \tag{2.9}$$

Definition 2.2 *a) We say that a continuous function* $u : \mathbb{R} \to E$ *is a classical solution of* (2.9)*, if* $u \in D(A) \cap D(B)$*,* $Bu \in C^1(R, E)$ *and* (2.9) *is satisfied.*

b) A continuous function $u : \mathbb{R} \to E$ *is called mild solution of* (2.9)*, if* $u \in D(B)$*,* $Bu(t) \in C(R, E)$*,* $\int_0^t u(s)ds \in D(A)$ *and*

$$Bu(t) = Bu(0) + A\int_0^t u(s)ds + \int_0^t f(s)ds \tag{2.10}$$

for all $t \in \mathbb{R}$*.*

Note that if $B = I$ and A is the generator of a C_0–semigroup $(T(t))$, then the mild solution of the form (2.10) coincides with the solution (1.3).

Lemma 2.3 *Let u be a mild solution of* (2.9)*. If Bu is in $C^1(R,E)$, then u is a classical solution.*

Proof. Let $u_h := \frac{1}{h}\int_t^{t+h} u(s)ds$, then $\lim\limits_{h\to 0} u_h = u(t)$. Moreover,

$$\begin{aligned} lim_{h\to 0}Au_h &= lim_{h\to 0}\frac{1}{h}A\int_t^{t+h} u(s)ds \\ &= lim_{h\to 0}\frac{1}{h}(A\int_0^{t+h} u(s)ds - A\int_0^t u(s)ds) \\ &= \frac{d}{dt}(A\int_0^t u(s)ds) \\ &= \frac{d}{dt}(Bu(t) - Bu(0) - \int_0^t f(t)) \\ &= \frac{d}{dt}Bu(t) - f(t), \end{aligned}$$

which exists. Since A is a closed operator, we obtain $u(t) \in D(A)$ and

$$Au(t) = \frac{d}{dt}Bu(t) + f(t)$$

for all $t \in \mathbb{R}$. Thus, u is a classical solution. ♣

The *Carleman transform* of a function $u \in L^\infty(\mathbb{R}, E)$ is defined by

$$\hat{u}(\lambda) = \begin{cases} \int_0^\infty e^{-\lambda t}u(t)dt, \ Re(\lambda) > 0 \\ -\int_{-\infty}^0 e^{-\lambda t}u(t)dt, \ Re(\lambda) < 0 \end{cases} \tag{2.11}$$

It is clear that $\hat{u}$ is holomorphic on $\mathbb{C} \setminus i\mathbb{R}$. A point $\mu \in \mathbb{R}$ is called a *regular point* of u if $\hat{u}$ has a holomorphic extension in a neighborhood of $i\mu$. The spectrum of u is defined as follows

$$sp(u) = \{\mu \in \mathbb{R} : \mu \text{ is not regular } \}.$$

It turns out (see [1]) that this spectrum coincides with the *Beurling spectrum*

$$sp(u) = \{\lambda \in \mathbb{R} : \forall \epsilon > 0\ \exists \phi \in L^1(\mathbb{R}) \text{ s.t. } supp\mathcal{F}\phi \subset (\lambda - \epsilon, \lambda + \epsilon) \text{ and } u * \phi \neq 0\},$$

where

$$\mathcal{F}\phi(t) = \int_{-\infty}^\infty e^{-ist}\phi(s)ds$$

is the Fourier transform of f. In the following lemma, we summarize some important properties of this spectrum, whose proof can be found in [10] and [17].

Lemma 2.4 *Let f, g be in $BUC(\mathbb{R}, E)$ and $\phi \in L^1(\mathbb{R}, E)$. Then*

- *(i) $sp(f)$ is closed and $sp(f) = \emptyset$ if and only if $f = 0$.*
- *(ii) $sp(f + g) \subset sp(f) + sp(g)$.*
- *(iii) $sp(f * \phi) \subset sp(f) \cap supp\mathcal{F}\phi$, where $\mathcal{F}\phi$ is the Fourier transform of ϕ.*

Taking now the Carleman transform on both sides of Equation (2.10) we obtain

$$B\hat{u}(\lambda) = \frac{1}{\lambda}Bu(0) + \frac{1}{\lambda}A\hat{u}(\lambda) + \frac{1}{\lambda}\hat{f}(\lambda),$$

so

$$(\lambda B - A)\hat{u}(\lambda) = Bu(0) + \hat{f}(\lambda)$$

for $\lambda \notin i\mathbb{R}$. Hence, for $\lambda \in \varrho(A, B)$ we have

$$\hat{u}(\lambda) = (\lambda B - A)^{-1}(Bu(0) + \hat{f}(\lambda)).$$

From Lemma 2.1(*iii*) this implies that, if $\mu \in R$ is a regular point of f and $i\mu \in \varrho(A, B)$ then $\hat{u}$ has a holomorphic extension in a neighborhood of $i\mu$, i.e. μ is a regular point of u. So, we have in fact proved the following

Theorem 2.5 *Let f be a bounded continuous function and u be a bounded mild solution of Equation* (2.9)*. Then the following holds.*

$$sp(u) \subseteq \{\mu \in R : i\mu \in \sigma(A, B)\} \cup sp(f).$$

From Theorem 2.5 we obtain the following corollaries.

Corollary 2.6 *If u is a bounded mild solution of Equation* (2.9) *corresponding to $f \equiv 0$, then $isp(u) \subseteq \sigma(A, B) \cap i\mathbb{R}$*

Corollary 2.7 *If $\sigma(A, B) \cap i\mathbb{R} = \emptyset$, then Equation* (2.9) *has at most one bounded mild solution.*

We close this section with the following lemma, which will be used later.

Lemma 2.8 *If u is a mild solution of Equation* (2.9) *corresponding to f, then $S(h)u$ is a mild solution of* (2.9) *corresponding to $S(h)f$, where $S(h)x(t) := x(t+h)$.*

Proof. Let u be a mild solution corresponding to f, then

$$\begin{aligned} BS(h)u(t) &= Bu(t+h) \\ &= Bu(0) + A\int_0^{t+h} u(s)ds + \int_0^{t+h} f(s)ds \\ &= \Big(Bu(0) + A\int_0^{h} u(s)ds + \int_0^{h} f(s)ds\Big) \\ &\qquad\qquad + (A\int_h^{t+h} u(s)ds + \int_h^{t+h} f(s)ds) \\ &= BS(h)u(0) + A\int_0^{t} S(h)u(s)ds + \int_0^{t} S(h)f(s)ds. \end{aligned}$$

Hence, $S(h)u$ is a mild solution corresponding to $S(h)f$. ♣

3 The Operator Equation $AX - BXD = C$

Let A and B be closed operators on a Banach space E, D be a closed, densely defined operator on Banach space F and C be a bounded operator from F to E. By $Dom(A)$ and $Range(A)$ we denote the domain and the range of any operator A, respectively. A bounded operator $X : F \to E$ is called a *bounded solution* of the operator equation

$$AX - BXD = C \tag{3.1}$$

if $Range(X) \subseteq Dom(B)$ and for each $f \in Dom(D)$, $Xf \in Dom(A)$, and $AXf - BXDf = Cf$.

A special case of Equation (3.1), where $B \equiv Id$, i.e.

$$AX - XD = C, \tag{3.2}$$

has been intensively considered by many authors (see [4, 9, 22, 24, 26] and the references therein). If B is bounded and invertible then we can convert

Equation (3.1) into Equation (3.2) by multiplying both sides of (3.1) by B^{-1}. If B is not invertible, then the situation is quite different, even when A or B, or both are bounded operators. Closely related to Equation (3.1) are the two operators $\tau_{A,B,D}$ and τ_B on $L(F,E)$ defined as follows.

(i) $Dom(\tau_{A,B,D}) = \{X \in L(F,E) : XF \subset Dom(B), XDom(D) \subset Dom(A)$ and $\exists Y \in L(F,E)$ such that $AXy - BXDy = Yy \; \forall y \in Dom(D)\}$

and

$$\tau_{A,B,D}X := Y;$$

(ii) $Dom(\tau_B) = \{X \in L(F,E) : Range(X) \subseteq Dom(B)\}$ and $\tau_B X := BX$.

To study these operators, we introduce the rank-one operators on $L(F,E)$ as follows: For any $x \in E$ and $\phi \in F'$, we define the bounded operator $X_{\phi,x} \in L(F,E)$ by

$$X_{\phi,x}(y) :=< y, \phi > x.$$

We have the following

Lemma 3.1 (cf. [2, Lemma 1.3]) *Let* $\phi \in F'\backslash\{0\}$ *and* $x \in E$ *such that* $Bx \neq 0$. *Then* $X_{\phi,x} \in Dom(\tau_{A,B,D})$ *if and only if* $x \in Dom(A) \cap Dom(B)$ *and* $\phi \in Dom(D')$. *In this case* $\tau_{A,B,D}(X_{\phi,x}) = X_{\phi,Ax} - X_{D'\phi,Bx}$.

Proof. Let $X_{\phi,x} \in Dom(\tau_{A,B,D})$. Then $X_{\phi,x}Dom(D) \in Dom(A)$ and $X_{\phi,x}F \in Dom(B)$. It follows that $x \in Dom(A)$ and $x \in Dom(B)$, i.e., $x \in Dom(A) \cap Dom(B)$. Let $Y = \tau_{A,B,D}X_{\phi,x}$, then, for each $y \in Dom(D)$, we have

$$Yy =< y, \phi > Ax- < Dy, \phi > Bx. \tag{3.3}$$

Let $\psi \in E'$ with $< Bx, \psi >= 1$. Using (3.3) we obtain

$$< Yy, \psi >=< y, \phi >< Ax, \psi > - < Dy, \phi >,$$

so

$$\begin{aligned} < Dy, \phi > &= < y, \phi >< Ax, \psi > - < Yy, \psi > \\ &= < y, < Ax, \psi > \phi - Y'\psi > \end{aligned}$$

for all $y \in Dom(D)$. It follows that $\phi \in Dom(D')$ and $D'\phi =< Ax, \psi > \phi - Y'\psi$.

Conversely, if $x \in Dom(A) \cap Dom(B)$ and $\phi \in Dom(D')$, then it is easy to see that $X_{\phi,x}F \subset Dom(B)$, $X_{\phi,x}Dom(D) \subset Dom(A)$. Hence,

$$\begin{aligned} \tau_{A,B,D}(X_{\phi,x})y &= AX_{\phi,x}y - BX_{\phi,x}Dy \\ &= < y, \phi > Ax - < Dy, \phi > Bx \\ &= < y, \phi > Ax - < y, D'\phi > Bx \\ &= X_{\phi,Ax}y - X_{D'\phi,Bx}y \end{aligned}$$

for every $y \in Dom(D)$. It follows that $\tau_{A,B,D}(X_{\phi,x})$ is a bounded operator from F to E, and hence, $X_{\phi,x} \in Dom(\tau_{A,B,D})$. ♣

Using Lemma 3.1, we obtain

Lemma 3.2 *i) The following holds*

$$\sigma_{ap}(A, B) - \sigma_{ap}(D') \subset \sigma(\tau_{A,B,D}, \tau_B);$$

ii) If $D(A) \cap D(B)$ is dense in E, then

$$\sigma_{ap}(A', B') - \sigma_{ap}(D) \subset \sigma(\tau_{A,B,D}, \tau_B).$$

Proof. Both parts of this Lemma will be proved by using the technique used in [2, Lemma 2.2]. We present here the proof of part *i)*. Part *ii)* can be proved with similar modification, and is ommited.

Let $\lambda \in \sigma_{ap}(A, B)$ and $\mu \in \sigma_{ap}(D')$. Then there exist $x_n \in Dom(A) \cap Dom(B)$ such that $\|Bx_n\| = 1$ and $(\lambda B - A)x_n \to 0$, and $\phi_n \in Dom(D')$ such that $\|\phi_n\| = 1$ and $(\mu - D')\phi_n \to 0$ for $n \to \infty$.

Define $X_n := X_{\phi_n,x_n}$; then, by Lemma 3.1, $X_n \in Dom(\tau_{A,B,D}) \cap Dom(\tau_B)$. Moreover, $\tau_B(X_n) = X_{\phi_n,Bx_n}$ and hence, $\|\tau_B X_n\| = \|Bx_n\| = 1$. Again, by Lemma 3.1, we have

$$\begin{aligned} \|[(\lambda - \mu)\tau_B - \tau_{A,B,D}](X_n)\| &= \|(\lambda - \mu)X_{\phi_n,Bx_n} - (X_{\phi_n,Ax_n} - X_{D'\phi_n,Bx_n})\| \\ &= \|X_{\phi_n,(\lambda B - A)x_n} - X_{(\mu - D')\phi_n,Bx_n}\| \\ &\square \|(\lambda B - A)x_n\| + \|(\mu - D')\phi_n\| \cdot \|Bx_n\| \to 0. \end{aligned}$$

Thus, $(\lambda - \mu) \in \sigma_{ap}(\tau_{A,B,D}, \tau_B) \subset \sigma(\tau_{A,B,D}, \tau_B)$. ♣

We now are in a position to state the main result of this section.

Theorem 3.3 *Assume that* $\sigma(A,B) \neq \mathbb{C}$ *or* $\sigma(D) \neq \mathbb{C}$*. Then*

$$\sigma(A,B) - \sigma(D) \subset \sigma(\tau_{A,B,D}, \tau_B). \tag{3.4}$$

For this we use the following lemma, whose proof can be found in [2, Lemma 2.3].

Lemma 3.4 *Let* N*,* M *be closed subsets of* $\mathbb{C}$ *such that* $M \neq \mathbb{C}$ *or* $N \neq \mathbb{C}$*. Then* $M - N \subset (\partial M - N) \cup (M - \partial N)$*.*

Proof of Theorem 3.3. By Lemma 2.1*ii*) we have $\partial\sigma(A,B) \subset \sigma_{ap}(A,B)$. Thus, by Lemma 3.2*i*,

$$\partial\sigma(A,B) - \sigma_{ap}(D') \subset \sigma_{ap}(A,B) - \sigma_{ap}(D') \subset \sigma(\tau_{A,B,D}, \tau_B),$$

and by Lemma 3.2,*ii*),

$$\begin{aligned} \partial\sigma(A,B) - \sigma_{ap}(D) &= \partial\sigma(A',B') - \sigma_{ap}(D) \\ &\subset \sigma_{ap}(A',B') - \sigma_{ap}(D) \subset \sigma(\tau_{A,B,D}, \tau_B). \end{aligned}$$

Hence,

$$\begin{aligned} \partial\sigma(A,B) - \sigma(D) &\subset \partial\sigma(A,B) - (\sigma_{ap}(D) \cup \sigma_{ap}(D')) \\ &= (\partial\sigma(A,B) - \sigma_{ap}(D)) \cup (\partial\sigma(A,B) - \sigma_{ap}(D')) \\ &\subset \sigma(\tau_{A,B,D}, \tau_B). \end{aligned}$$

Similarly, we have $\sigma(A,B) - \partial\sigma(D) \subset \sigma(\tau_{A,B,D}, \tau_B)$. Using Lemma 3.4, we obtain $\sigma(A,B) - \sigma(D) \subset \sigma(\tau_{A,B,D}, \tau_B)$, and the theorem is proved. ♣

In particular, Theorem 3.3 implies

Corollary 3.5 *If for every bounded operator* $C : F \mapsto E$*, the equation*

$$AX - BXD = C$$

has a unique bounded solution, then $\sigma(A,B) \cap \sigma(D) = \emptyset$*.*

Proof. From the assumption it follows that $0 \in \varrho(\tau_{A,B,D})$ and thus, $0 \in \varrho(\tau_{A,B,D}, \tau_B)$. By Theorem 3.3, it implies $0 \notin \sigma(A, B) - \sigma(D)$, i.e. $\sigma(A, B) \cap \sigma(D) = \emptyset$. ♣

The converse of Corollary 3.5 is generally false, even for the case when $B = I$ (see [22, Example 9]). However, it holds in some particular cases. For example, when $B = I$, and A and D generate C_0-semigroups, one of which is eventually norm continuous (see [2]), and the following case.

Theorem 3.6 *Let A, B be closed operators with $Dom(A) \cap Dom(B)$ dense in E, and D be a bounded operator. If the spectra $\sigma(A, B)$ and $\sigma(D)$ are disjoint, then, for every bounded operator C, Equation* (2.9) *has a unique bounded solution. More precisely, if Γ is a Cauchy contour around $\sigma(D)$ and separated from $\sigma(A, B)$, then*

$$X = -\frac{1}{2\pi i} \int_\Gamma (\lambda B - A)^{-1} C (\lambda - D)^{-1} d\lambda. \tag{3.5}$$

Proof. We first show that X defined by (3.5) is a solution of Equation (3.1). To this end, let X be given by (3.5). Then it is easy to see the following:

(*i*) $Xf \in Dom(A)$ for every $f \in F$ and

$$AXf = -\frac{1}{2\pi i} \int_\Gamma A(\lambda B - A)^{-1} C (\lambda - D)^{-1} f d\lambda;$$

(*ii*) $Xf \in Dom(B)$ for every $f \in F$ and

$$BXf = -\frac{1}{2\pi i} \int_\Gamma B(\lambda B - A)^{-1} C (\lambda - D)^{-1} f d\lambda;$$

(*iii*) $\int_\Gamma (\lambda B - A)^{-1} d\lambda = 0$ and thus, $\int_\Gamma B(\lambda B - A)^{-1} d\lambda = 0$.

Let now $f \in Dom(D)$. Using (*i*), (*ii*) and (*iii*) we obtain

$$\begin{aligned}
AXf &= -\frac{1}{2\pi i} \int_\Gamma A(\lambda B - A)^{-1} C (\lambda - D)^{-1} f d\lambda \\
&= -\frac{1}{2\pi i} \int_\Gamma \{(\lambda B - (\lambda B - A)\} (\lambda B - A)^{-1} C (\lambda - D)^{-1} f d\lambda \\
&= -\frac{1}{2\pi i} \int_\Gamma \lambda B(\lambda B - A)^{-1} C (\lambda - D)^{-1} f d\lambda + \frac{1}{2\pi i} \int_\Gamma C (\lambda - D)^{-1} f d\lambda \\
&= -\frac{1}{2i\pi} \int_\Gamma B(\lambda B - A)^{-1} C [(\lambda - D) + D] (\lambda - D)^{-1} f d\lambda + Cf
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2i\pi}\int_\Gamma B(\lambda B - A)^{-1} C f d\lambda - \frac{1}{2i\pi}\int_\Gamma B(\lambda B - A)^{-1} C(\lambda - D)^{-1} D f d\lambda + Cf \\
&= BXDf + Cf.
\end{aligned}
$$

Hence, X is a solution of (3.1). Conversely, if X is a solution of (3.1). Then

$$C = BX(\lambda - D) - (\lambda B - A)X.$$

It follows that

$$(\lambda B - A)^{-1} C(\lambda - D)^{-1} = (\lambda B - A)^{-1} BX - X(\lambda - D)^{-1}.$$

Hence,

$$
\begin{aligned}
\frac{1}{2\pi i}\int_\Gamma (\lambda B - A)^{-1} C(\lambda - D)^{-1} d\lambda &= \frac{1}{2\pi i}\int_\Gamma (\lambda B - A)^{-1} BX d\lambda \\
&\qquad - \frac{1}{2\pi i}\int_\Gamma X(\lambda - D)^{-1} d\lambda \\
&= -X,
\end{aligned}
$$

and the theorem is proved ♣

4 Regularly Admissible Subspaces

A closed, translation-invariant subspace $\mathcal{M}$ of $BUC(R, E)$ is said to be regularly admissible with respect to equation

$$\frac{d}{dt}Bu(t) = Au(t) + f(t). \tag{4.1}$$

if

(*i*) *for every* $f \in \mathcal{M}$*, there exists a unique mild solution* $u \in \mathcal{M}$ *of* (4.1) *and*

(*ii*) $f_n \to f$ *in* $\mathcal{M}$ *implies* $u_n \to u$*, where* u *and* u_n *are the mild solution of* (4.1) *corresponding to* f *and* f_n*, respectively.*

If $\mathcal{M}$ is a regularly admissible, we define the operator $G : \mathcal{M} \to \mathcal{M}$ as follows: Let $f \in \mathcal{M}$, then Gf is the mild solution of (4.1) corresponding to f. From (*ii*) it follows that G is a bounded operator. G is called *the solution operator* of Equation (4.1).

Lemma 4.1 *Suppose $\mathcal{M}$ is a regularly admissible subset of $BUC(R,E)$ and $f \in \mathcal{M}$ with $f' \in \mathcal{M}$. Then u, the mild solution corresponding to f, is differentiable and u' is the mild solution corresponding to $f'(t)$.*

Proof. Let $f \in \mathcal{M}$ with $f' \in \mathcal{M}$ and u be the mild solution corresponding to f. By Lemma 2.8 we have $GS(h)f = S(h)Gf$, where $S(h)$ is the translation operator on $\mathcal{M}$. Hence, we have

$$\begin{aligned} lim_{h\to 0}\frac{u(t+h)-u(t)}{h} &= lim_{h\to 0}\frac{(S(h)Gf-Gf)}{h}(t) \\ &= lim_{h\to 0}\frac{G(S(h)f-f)}{h}(t) \\ &= (Glim_{h\to 0}\frac{(S(h)f-f)}{h})(t) \\ &= (Gf')(t). \end{aligned}$$

Thus, u is differentiable and u' is the mild solution corresponding to f'. ♣

In what follows, we assume that $\mathcal{M}$ satisfies the following additional assumption:

$$\textit{For all } C \in \mathcal{L}(\mathcal{M},E) \textit{ and } f \in \mathcal{M}, \tag{4.2}$$

$$\textit{the function } \Phi(t) = CS(t)f \textit{ belongs to } \mathcal{M}.$$

The admissibility of a space is closely related to the solvability of the operator equation (3.1), as the following Theorem shows.

Theorem 4.2 *Let A be a linear, closed, densely defined operator and B be a bounded operator on E, and let $\mathcal{M}$ be a translation invariant subspace in $BUC(R,E)$ satisfying condition (4.2). Let $\delta_0 : \mathcal{M} \to E$ be defined by $\delta_0(f) = f(0)$ and $\mathcal{D}_{\mathcal{M}} : Dom(\mathcal{D}_{\mathcal{M}}) \to \mathcal{M}$ by:*

$$Dom(\mathcal{D}_{\mathcal{M}}) = \{f \in \mathcal{M} : f' \in \mathcal{M}\} \textit{ and } \mathcal{D}_{\mathcal{M}}f = f'.$$

Then the following statements are equivalent.

(i) $\mathcal{M}$ is a regularly admissible subspace w.r.t. (4.1).

(ii) The operator equation

$$AX - BX\mathcal{D}_{\mathcal{M}} = -\delta_0 \tag{4.3}$$

has a unique bounded solution.

(iii) For every bounded operator $C : \mathcal{M} \to E$, the operator equation

$$AX - BX\mathcal{D}_{\mathcal{M}} = C \tag{4.4}$$

has a unique solution.

Proof (*i*) $\Rightarrow$ (*ii*). Let $G : \mathcal{M} \to \mathcal{M}$ be the solution operator, i.e. Gf is the unique mild solution of (4.1) in $\mathcal{M}$. Define the bounded operator $X : \mathcal{M} \mapsto E$ by $Xf := (Gf)(0)$. If $f \in Dom(\mathcal{D}_{\mathcal{M}})$, then, by Lemma 4.1, $u = Gf$ is a classical solution to (2.9) and

$$BG(f')(t) = \frac{d}{dt}B(Gf)(t) = A(Gf)(t) + f(t). \tag{4.5}$$

Taking $t = 0$ from (4.5), we obtain $AXf - BX\mathcal{D}_{\mathcal{M}}f = -\delta_0 f$. So, X is a bounded solution of (4.3).

Conversely, let X be any bounded solution of (4.3) and $f \in Dom(\mathcal{D}_{\mathcal{M}})$. Defining $u(t) := XS(t)f$, then $u \in \mathcal{M}$ and

$$\frac{d}{dt}Bu(t) = BX\mathcal{D}_{\mathcal{M}}S(t)f = (AX + \delta_0)S(t)f = Au(t) + f(t),$$

i.e. u is a classical, and thus, a mild solution of Equation (4.1).

To show the uniqueness of X, assume that there is another bounded solution of equation $AX - BX\mathcal{D}_{\mathcal{M}} = -\delta_0$, say X_0. Define $u_0(t) = X_0S(t)f$, then u_0 is another mild solution of (4.1) in $\mathcal{M}$. By the uniqueness of the solution of this equation, we have $u \equiv u_0$, which implies $X = X_0$.

(*ii*) $\Rightarrow$ (*iii*) Let X be the unique solution of (4.3). We define the bounded operator $Y : \mathcal{M} \to E$ by $Yf := X\overline{f}$, where $\overline{f}(t) = -CS(t)f$. If $f \in Dom(\mathcal{D}_{\mathcal{M}})$, then $\overline{(\mathcal{D}_{\mathcal{M}}f)}(t) = -CS(t)\mathcal{D}_{\mathcal{M}}f = \mathcal{D}_{\mathcal{M}}\overline{f}(t)$. Hence,

$$AYf = AX\overline{f} = (BX\mathcal{D}_{\mathcal{M}} - \delta_0)\overline{f} = BX\overline{\mathcal{D}_{\mathcal{M}}f} + Cf = BY\mathcal{D}_{\mathcal{M}}f + Cf,$$

i.e. Y is a bounded solution of (4.4).

The uniqueness of the solution of operator equation $AX - BX\mathcal{D}_{\mathcal{M}} = C$ follows directly from that of the solution of $AX - BX\mathcal{D}_{\mathcal{M}} = -\delta_0$.

$(iii) \Rightarrow (i)$ We have shown above that, if X is a bounded solution of (4.3) and $f \in Dom(\mathcal{D}_{\mathcal{M}})$, then $u(t) := XS(t)f$ is a mild solution of Equation (4.1). We will show this is also true for every $f \in \mathcal{M}$. Let $f \in \mathcal{M}$. Since $Dom(\mathcal{D}_{\mathcal{M}})$ is dense in $\mathcal{M}$, there exists a sequence $\{f_n\} \subset Dom(\mathcal{D}_{\mathcal{M}})$ such that $f_n \to f$ in $\mathcal{M}$. Put $u_n(t) := XS(t)f_n$, then $u_n(t) \to u(t) = XS(t)f$ in $\mathcal{M}$. Hence $\int_0^t u_n(\tau)d\tau \to \int_0^t u(\tau)d\tau$ and

$$\begin{aligned} A\int_0^t u_n(s)ds &= Bu_n(t) - Bu_n(0) - \int_0^t f_n(s)ds \\ &\longrightarrow Bu(t) - Bu(0) - \int_0^t f(s)ds \end{aligned}$$

as $n \to \infty$. Since A is closed, we have $\int_0^t u(s)ds \in Dom(A)$ and

$$A\int_0^t u(s)ds = Bu(t) - Bu(0) - \int_0^t f(s)ds,$$

i.e. u is a mild solution to (4.1).

Next, we show that $u(t) = XS(t)f$ is the unique solution to (4.1). To this end, assume that (4.1) has another mild solution, say u_0. Put $v := u - u_0$, then v is a mild solution of the homogeneous equation $\frac{d}{dt}Bu(t) = Au(t)$, $t \in \mathbb{R}$. By Corollary 2.6, $isp(v) \subseteq \sigma(A,B)$. On the other hand, since $v \in \mathcal{M}$, $isp(v) \subseteq \sigma(\mathcal{D}_{\mathcal{M}})$ ([23, Proposition 3.4]. So, $isp(v) \subseteq \sigma(A,B) \cap \sigma(\mathcal{D}_{\mathcal{M}})$. By Corollary 3.5, statement (iii) implies that $\sigma(A,B) \cap \sigma(\mathcal{D}_{\mathcal{M}}) = \emptyset$. Hence, we have $sp(v) = \emptyset$, so $v \equiv 0$.

Finally, if $f_n \to f$ in $\mathcal{M}$, then, from the identities $u_n(t) = XS(t)f_n$ and $u(t) = XS(t)f$, it is easy to see that $u_n \to u$ in $\mathcal{M}$. Hence, $\mathcal{M}$ is regularly admissible , and the theorem is complete. ♣

From Theorem 4.2, it follows

Corollary 4.3 *Suppose A is a linear, closed, densely defined operator and B is a bounded operator on a Banach space E, and Δ is a subset of* $\mathbb{R}$. *Let $X(\Delta)$ denote the subspace of $BUC(\mathbb{R}, E)$ which contains all functions f with $sp(f) \subset \Delta$.*

(i) If $X(\Delta)$ is regularly admissible w.r.t. (4.1), *then $i\Delta \subseteq \varrho(A,B)$ and there is a constant C such that*

$$\|(i\lambda B - A)^{-1}\| \square C \text{ for all } \lambda \in \Delta; \tag{4.6}$$

(ii) If Δ is compact, then $X(\Delta)$ is regularly admissible w.r.t. (4.1) *if and only if $i\Delta \subseteq \varrho(A,B)$.*

Proof. (i): It is well-known that $\sigma(\mathcal{D}_{X(\Delta)}) = i\Delta$ (see e.g. [1]). Assuming that $X(\Delta)$ is regularly admissible w.r.t. (4.1), then, by Theorem 4.2, equation $AX - BX\mathcal{D}_{X(\Delta)} = C$ has a unique bounded solution for every bounded C. Hence, by Corollary 3.5, we have $\sigma(A,B) \cap \Delta = \sigma(A,B) \cap \sigma(\mathcal{D}_{X(\Delta)}) = \emptyset$, or, equivalently, $i\Delta \subseteq \varrho(A,B)$.

Let G be the solution operator and x_0 be any vector in E. Put $f(t) := e^{i\lambda t}x_0$, where $\lambda \in \Delta$; then it is not hard to see that $f \in X(\Delta)$ and $(Gf)(t) = e^{i\lambda t}y_0$, where $y_0 = (i\lambda B - A)^{-1}x_0$. Hence,

$$\|(i\lambda B - A)^{-1}x_0\| = \|Gf\| \square \|G\| \cdot \|f\| = \|G\| \cdot \|x_0\|$$

for all $x_0 \in E$ and $\lambda \in \Delta$. Hence $\|(i\lambda B - A)^{-1}\| \square \|G\| < \infty$.

(ii) If Δ is compact, then $\mathcal{D}_{X(\Delta)}$ is a bounded operator on $X(\Delta)$ (see [5]). So, from Theorem 3.6 it follows that equation $AX - BX\mathcal{D}_{X(\Delta)} = C$ has a unique bounded solution for every bounded C. By Theorem 4.2, $X(\Delta)$ is regularly admissible w.r.t. (4.1). $\clubsuit$

We now apply Corollary 4.3 to $P(\omega)$, the space of all continuous, periodic functions with period ω. It is not hard to see that $P(\omega) = X(\Delta)$, where $\Delta = \{\frac{2k\pi}{\omega} : k \in \mathbb{Z}\}$. Hence, by Corollary 4.3, if $P(\omega)$ is regularly admissible w.r.t. (4.1), then

$$\frac{2\pi ik}{\omega} \in \varrho(A,B) \text{ for all } k \in \mathbb{Z}$$

and

$$\sup_{k\in\mathbb{Z}} \|(\frac{2\pi ik}{\omega}B - A)^{-1}\| < \infty.$$

5 Mild solutions in subspaces of Λ–class

In this section we use the results obtained in the previous sections to investigate the case, in which the subspace $\mathcal{M}$ is generally not regularly admissible w.r.t. (4.1) in the sense that Equation (4.1) can not be solved uniquely. In this situation, we will find conditions on A and B such that every bounded mild solution of (4.1) will belong to $\mathcal{M}$. For the case when $B = I$, this problem has recently been of great interest (see [3, 18], when A is generator of a C_0 semigroup and [1, 13, 14], when A is a closed operator). Using the results obtained in Section 3 and 4, we will prove e.g. that under some classical condition, every bounded mild solution of Equation (4.1) is (weakly) (asymptotically) almost periodic whenever f is so, provided $\sigma(A,B) \cap i\mathbb{R}$ is countable. The proofs are quite short. Note that the method of sums of commuting operators introduced in [13] can not apply to our situation, since A and B are not commuting. We begin with the definition of a Λ–class subspace.

Definition 5.1 (cf. [3, Definition 2.1]) *A closed, translation invariant subspace $\mathcal{M} \subset BUC(\mathbb{R}, E)$ is said to be a Λ–class if it satisfies*

(i) $\mathcal{M}$ contains all the constant functions;

(ii) It $u \in \mathcal{M}$ and $\lambda \in \mathbb{R}$ then $u_\lambda \in \mathcal{M}$, where $u_\lambda(t) = e^{i\lambda t}u(t)$.

(iii) If $u \in \mathcal{M}$ and $B \in L(E,E)$, then $Bu \in \mathcal{M}$, where $(Bu)(t) = Bu(t)$.

Typical examples of Λ–classes are the following spaces (for the definition of these spaces see e.g. [11, 18]).

+) $AP(\mathbb{R}, E)$, the space of almost periodic functions.

+) $AAP(\mathbb{R}, E)$, the space of asymptotically almost periodic functions.

+) $WAAP(\mathbb{R}, E)$, the space of weakly asymptotically almost periodic functions.

Let $\mathcal{M}$ be a Λ–class and $u \in BUC(\mathbb{R}, E)$. We say that the number $\lambda \in \mathbb{R}$

is a $\mathcal{M}$–*regular* point of u, if there exists a number $\epsilon > 0$, such that for every function $\phi \in L^1(\mathbb{R})$ with $\mathrm{supp}\mathcal{F}\phi \subset [\lambda - \epsilon, \lambda + \epsilon]$, $\phi * u \in \mathcal{M}$. The complement set in $\mathbb{R}$ of all $\mathcal{M}$–regular points is called the $\mathcal{M}$–*spectrum* of u, and denoted by $sp_{\mathcal{M}}(u)$.

We now return to Equation (4.1) and assume that A is a closed, densely defined and B a bounded operator on E. Let Δ be a compact set in $\mathbb{R}$ and $X(\Delta)$ the subspace of $BUC(\mathbb{R}, E)$ consisting of all functions f with $sp(f) \subset \Delta$. It is easy to see that $X(\Delta)$ satisfies condition (4.2). By Corollary 4.3, $X(\Delta)$ is regularly admissible w.r.t. (4.1) and, by Theorem 3.6, for any function $f \in X(\Delta)$ the mild solution u is obtained by

$$u(\cdot) = XS(\cdot)f = \frac{1}{2i\pi}\int_\Gamma (\lambda B - A)^{-1}\delta_0 S(\cdot)(\lambda - D_{X(\Gamma)})^{-1} f d\lambda, \tag{5.1}$$

where Γ is the Cauchy contour around Δ and separated from $\sigma(A, B)$.

Lemma 5.2 *Let u be defined by* (5.1) *and $\mathcal{M}$ be any Λ–class. Then $u \in \mathcal{M}$ whenever $f \in \mathcal{M}$.*

Proof. Given $f \in BUC(\mathbb{R}, E)$, then the function $P : \Delta \to BUC(\mathbb{R}, E)$ defined by

$$P(\lambda) := (\lambda B - A)^{-1}\delta_0 S(\cdot)(\lambda - D_{X(\Delta)})^{-1} f$$

is continuous. Moreover, $(\lambda - D_{X(\Delta)})^{-1} f \in \mathcal{M}$ whenever $f \in \mathcal{M}$. Hence, from the definition of P, $P(\lambda)(t) = (\lambda B - A)^{-1}\Big((\lambda - D_{X(\Delta)})^{-1} f\Big)(t)$, we obtain $P(\lambda) \in \mathcal{M}$ (Definition 5.1 iii). Because $\mathcal{M}$ is a closed subspace, $u = 1/(2\pi i)\int_\Gamma P(\lambda) d\lambda$ is also in $\mathcal{M}$. ♣

We now are in a position to state the main result of this section.

Theorem 5.3 *If $\mathcal{M}$ is a Λ–class and $f \in \mathcal{M}$, then for every bounded mild solution u of Equation* (4.1)*, we have*

$$sp_{\mathcal{M}}(u) \subset \sigma(A, B) \cap i\mathbb{R}$$

Proof. It suffices to show that, if $\lambda \in \mathbb{R}$ and $i\lambda \notin \sigma(A, B)$, then λ is a $\mathcal{M}$–regular point of u. Since $\sigma(A, B)$ is closed, there is a number $\epsilon > 0$, such that $(i\Delta) \cap \sigma(A, B) = \emptyset$, where $\Delta := [\lambda - \epsilon, \lambda + \epsilon]$. By the arguments before this theorem, $X(\Delta)$ is regularly admissible w.r.t. (4.1).

Let $\phi \in L^1(\mathbb{R}, E)$ be any function with $\mathrm{supp}\mathcal{F}\phi \subset \Delta$, then $\tilde{u} := u * \phi$ and $\tilde{f} := f * \phi$ are in $X(\Delta)$ (Lemma 2.4(iii)). Moreover, $\tilde{f} \in \mathcal{M}$ ([3, pp 60]). Finally, it is not hard to show that $\tilde{u}$ is a bounded mild solution of equation

$$\frac{d}{dt}Bu(t) = Au(t) + \tilde{f}(t).$$

Hence, by Lemma 5.2, $\tilde{u}$ is also in $\mathcal{M}$. So, by definition, λ is a $\mathcal{M}$–regular point of u, and the theorem is proved. ♣

Let u be a bounded mild solution of Equation (4.1) and suppose $\sigma(A,B) \cap i\mathbb{R}$ is countable. By Theorem 5.3, $sp_{\mathcal{M}}(u)$ countable, too. With this countability of $sp_{\mathcal{M}}(u)$, we can reveal some properties of u:

i) ([3, Theorem 2.6]) For any Λ-class $\mathcal{M}$, u will belong to $\mathcal{M}$ if for every $\lambda \in sp_{\mathcal{M}}(u)$,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t e^{-i\lambda s}u(x+s)ds$$

exists uniformly w.r.t. $x \in \mathbb{R}$.

ii) (Loomis' Theorem, [11, Theorem 6.4]) For $\mathcal{M} = AP(\mathbb{R}, E)$, u is almost periodic if either (a) $E \not\supset c_0$ or (b) The range of $u(t)$ is weakly relatively compact.

Applying Theorem 5.3 and the above cited results, we obtain

Corollary 5.4 *Assume that $\sigma(A,B) \cap i\mathbb{R}$ is countable and f is an almost periodic function. Then every bounded mild solution u to (4.1) is almost periodic if one of the following conditions is satisfied.*

(a) $E \not\supset c_0$; or

(b) The range of $u(t)$ is weakly relatively compact; or

(c) For every $\lambda \in sp_{AP(\mathbb{R},E)}(u)$,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t e^{-i\lambda s}u(x+s)ds$$

exists uniformly w.r.t. $x \in \mathbb{R}$.

Corollary 5.5 *Assume that* $\sigma(A,B)\cap i\mathbb{R}$ *is countable and* f *is an asymptotically (weakly asymptotically) almost periodic function, then every bounded mild solution to* (4.1) *having the additional property that, for every* $\lambda \in sp_{AAP(\mathbb{R},E)}(u)$ *(for every* $\lambda \in sp_{WAAP(\mathbb{R},E)}(u)$*),*

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t e^{-i\lambda s}u(x+s)ds$$

exists uniformly w.r.t. $x \in \mathbb{R}$*, is asymptotically (weakly asymptotically) almost periodic.*

In case $B = I$, Corollary 5.4 and 5.5 have been proved in [18], where A is the generator of a C_0–semigroup, and Corollary 5.4 was proved in [1] for a closed operator A.

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