TAPE SPOOL EXPLORATIONS

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Article 00.: Introduction. Imagine a *Spool of Tape* as in *Figure_01 (below). Figure_01* illustrates a *full spool* of tape along with illustrations of: *ONE, SINGLE typical tape ring LAYER* which is highlighted with the *color NAVY BLUE*; the *SPOOL CORE radius = c* which is highlighted with the color *GREEN*; and, ... the *OUTER MOST TAPE RING Layer Radius = R* which is highlighted with the color *RED*. In addition to the illustrated quantity *symbols*, *R* and *c*, symbols for the *Thickness* of the tape and for the *Length* of the tape are also (here below) displayed. Hence,

c = *Core radius*
R = *outer most ring layer Radius*
$\boldsymbol{t}$ = **tape Thickness**
L = *tape Length*

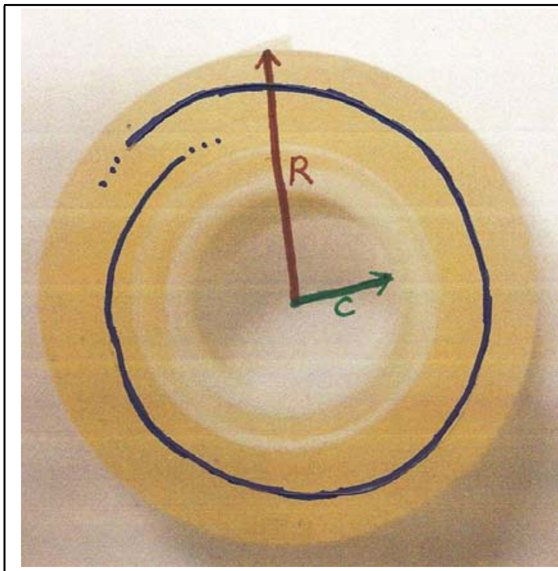

FIG_01: "ONE" typical tape ring.
c = *Core radius;* R = *Outer ring radius*

Now suppose that one wishes to *formulate* an algebraic relationship among these four quantities. There are, in fact, several undergraduate-level modeling schemes which accomplish such algebraic relationship; further, each such model is a clever application of undergraduate courses content.

Article 01.: Geometric Area Model. Applying this model, imagine equating *two different area formulations* of a single, area reference. Hence, imagine completely unrolling the entire tape spool; now, while holding that long length of tape horizontally, and while focusing on the *tape length EDGE*, the *AREA of that RECTANGULAR EDGE* is, of course:

(1.1) $L \cdot t = (\textit{Length of TAPE}) \times (\textit{Thickness of TAPE}) = \textit{Area of rectangular EDGE.}$

Now, observe that *winding that rectangular edge* back around the spool core creates the *Annular Area* bounded by the *core circumference* and the *outer-most ring layer* of the tape roll. For *very* *small values of* t, the *Area of the rectangular EDGE* has simply experienced an *Area Preserving Deformation into an Annular Area (for all practical purposes).* Further, that *Annular Area* is described by:

$$(1.2)\quad \textit{Annular Area} = \pi R^2 - \pi c^2 = \pi(R^2 - c^2)\,.$$

So, ... since (*for all practical purposes) these two areas are equal for small values of t,* we declare that:

$$(1.3)\quad \boxed{L \cdot t = \pi(R^2 - c^2)}\,.$$

Hence, given a spool of thin tape, Item (1.3) does present an algebraic relationship among those four quantities (for all practical purposes); and, the objective has been accomplished.

Therefore, the *Length of (thin) Tape,* L, wound around the spool, can be very closely estimated by the formula:

$$(1.4)\quad \boxed{L = \frac{\pi}{t}\,(R^2 - c^2)}\,.$$

This particular model is only one of several *geometric models* which can cleverly accomplish said objective.

Article 02.: Arithmetic Sequence Model. Applying this model, imagine the full spool of tape as a collection of nested, concentric ring layers. Hence, the *first tape ring layer* would simply be the *first, full tape wrap* around the spool core. The *second tape ring layer* would, of course, simply be the *second, full tape wrap.* Now, suppose that the *entire tape Length,* L, is wrapped around the core after a *number, say* n, such tape ring layers. So, the *value of* L is simply the *sum* of the *number,* n, of tape ring circumferences.

The *radius,* r_1, *must now be decided.* One choice is to use the *radius,* $r_1 = c$, *of the Bottom side* of that ring. Another choice would be to use the *radius,* $r_1 = (c + t)$, *of the Top side* of that ring. Or, one could choose that first layer radius to be the *intermediate value,* $r_1 = (c + \frac{1}{2}t)$. Supposing that the *mean value* of the extreme radii values would be preferable, we *Adopt the Model* which uses the *mean radii values.* Hence, for this Model, we declare that:

$$(2.1)\quad r_1 = (c + \tfrac{1}{2}t);\ \ r_2 = (c + \tfrac{3}{2}t);\ \ r_3 = (c + \tfrac{5}{2}t);\ \ \ldots\text{, and }\ r_n = (c + \tfrac{2n-1}{2}t)\,.$$

Note that Item (2.1) displays an *Arithmetic Sequence* with *common difference = t ,* since the tape *mid-thickness levels* are separated by the thickness value t. Therefore, the *Length, L,* is given by

$$(2.2)\quad L = \sum_{k=1}^{n} 2\pi\, r_k = 2\pi \sum_{k=1}^{n} r_k\,.$$

(2.3) At this point, we briefly digress to observe a property of *Arithmetic Sequences.* So, imagine an arithmetic sequence: F, $F + d$, $F + 2d$, ..., $F + (n-1)d$;
Note the *first term value,* F, and the *last term value,* $L = F + (n-1)d$.
Now, *observe* about the sum, S : $\sum_{k=0}^{n-1}(F + kd) = S = \sum_{k=0}^{n-1}(L - kd)\,.$

Thus, $2S \ = \ \sum_{k=0}^{n-1} [\,(F+kd)+(L-kd)\,] \ = \ \sum_{k=0}^{n-1} [\,F+L\,] \ = \ n\,(F+L).$

And, so: $\boxed{S \ = n \cdot [\,\frac{F+L}{2}\,]}$. Thus: *Sum = (number of terms)* × *(Mean of First & Last terms).*

Now, appealing to the *digression (2.3),*

$$(2.4) \quad \sum_{k=1}^{n} r_k \ = n \cdot \left[\frac{r_1 + r_n}{2}\right] \ = \ n \cdot \left[\frac{(c+\frac{1}{2}t)+(c+\frac{2n-1}{2}t)}{2}\right] \ = \ \frac{n}{2}\cdot[2c+nt]\,.$$

Referring to *Figure_01*, it follows that, $nt \ = \ (R-c)\,;$ using this equality in (2.4), we see that

$$(2.5) \quad \sum_{k=1}^{n} r_k \ = \ \frac{(R-c)}{2t}\,[\,2c+(R-c)\,] \ = \ \frac{(R-c)}{2t}\,(R+c) \ = \ \frac{(R^2-c^2)}{2t}\,.$$

Therefore, from (2.2) and (2.5), we have that

$$(2.6) \quad L \ = \ \sum_{k=1}^{n} 2\pi\, r_k \ = \ 2\pi \sum_{k=1}^{n} r_k \ = \ 2\pi \cdot \frac{(R^2-c^2)}{2t} \ = \ \boxed{\frac{\pi}{t}\cdot(R^2-c^2)}\,.$$

Hence, from (1.4) and (2.6), we see that *both Model_01 & Model_02* yield the same formulations.

<u>Article 03.: Archimedean Spiral Model</u>. Applying this model, imagine the full spool of tape as a *very tightly wrapped Archimedean Spiral. Figure_02, below,* illustrates loosely wrapped *red tape* with *widely spread spiral layer rings* surrounding a *green spool core.*

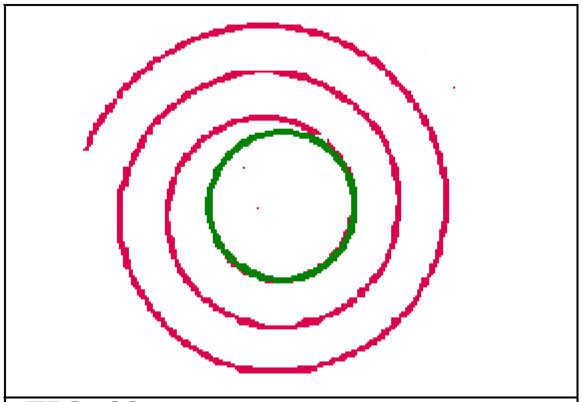

FIG_02: Green tape-Core radius $= c$
Loosely wrapped Tape, Red Spiral Layers

An *Archimedean Spiral* is defined by the *polar equation:* $r = A\theta$. Thus, applying this model, the coefficient, A, must be determined so that the tape roll is tightly wrapped. Since each successive, wrapped layer of tape should increase the r-*values* by the *tape thickness,* t, then

$$(3.1) \quad t \ = \ r(\theta+2\pi)-r(\theta) \ = \ A\cdot(\theta+2\pi)-A\cdot\theta \ = \ 2\pi A\,.$$

Hence, it follows that: $A = \frac{t}{2\pi}$; and, so

(3.2) $r(\theta) = \left(\frac{t}{2\pi}\right)\theta$.

Let θ_c and θ_R denote the respective *θ-values* for the *core radius* $= c$ and *outer ring radius* $= R$.

Then, we have that

(3.3) $c = r(\theta_c) = \left(\frac{t}{2\pi}\right)\theta_c$ and $R = r(\theta_R) = \left(\frac{t}{2\pi}\right)\theta_R$,

and so

(3.4) $\theta_c = \left(\frac{2\pi}{t}\right) c$ and $\theta_R = \left(\frac{2\pi}{t}\right) R$.

Next, we recall an *Arc-Length Formula* from first year Calculus;

(3.5) $$dL = \sqrt{\left[\frac{dr}{d\theta}\right]^2 + r^2}\ d\theta = \sqrt{\left(\frac{t}{2\pi}\right)^2 + \left(\frac{t}{2\pi}\theta\right)^2}\ d\theta = \frac{t}{2\pi}\sqrt{1+\theta^2}\ d\theta .$$

Now, we see that

(3.6) $$L = \frac{t}{2\pi}\int_{\theta_c}^{\theta_R} \sqrt{1+\theta^2}\ d\theta = \boxed{\int_c^R \sqrt{1+\left(\frac{2\pi}{t}u\right)^2}\ du},$$

after applying the *change of variable substitution:* $\theta = \frac{2\pi}{t} u$.

(3.7) OBSERVATIONAL NOTE: Again, by appealing to the *underlying assumption* that the *tape thickness* $= t$ *is very small (i.e., t <<1.0)* , we conclude that $\sqrt{1+(\frac{2\pi}{t}u)^2} \approx \frac{2\pi}{t}u$, $(c \le u \le R)$. And, using this approximation along with (3.6), we have

(3.8) $$L = \int_c^R \sqrt{1+\left(\frac{2\pi}{t}u\right)^2}\ du \approx \int_c^R \frac{2\pi}{t}u\ du = \frac{\pi}{t}u^2 \Big|_c^R = \boxed{\frac{\pi}{t}\cdot(R^2-c^2)}$$

.

Hence, by appealing to the assumption of a *thin value for tape thickness*, we note that Items (1.3), (2.6), and (3.8) declare that *ALL three Models yield the same relationships (and formulations for L)* .

Article 04.: Exploring and Investigating Other Models**.** What is *interesting, novel, and intriguing* about this *Tape Spool Exploration* is not only the above presented, cleverly applied models; but, further, that the *above list does NOT exhaust* the list of *analytical models*. So, ... *just for fun, ... explore further clever modeling notions* regarding this very *Tape Spool Exploration.*

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(* * * TAPE  MODELS  CALCULATIONS * * *)

(* Sample computations, below, illustrate that the AreaModel & SpiralModel
   applications render virtually the same calculated results.
    IF 2748 inches of tape were Wrapped around a Core of Radius c = 1.5
     inches AND the Tape OUTER RING Layer had Radius  R = 2.0 inches,
     THEN  ... the Tape THICKNESS t = ?                              *)

Clear[L, R, c, t]

AreaModel[L_, R_, c_, t_] := t * L - π (R^2 - c^2)

R =  2.0 ;
c = 1.5  ;
(*  t= ?    ;     *)
L = 2748 ;

Solve[AreaModel[L, R, c, t] == 0, t]

{{t → 0.0020006503434432816`}}
(*  Hence, the TAPE THICKNESS calculates to t = 0.002 (TWO THOUSANDTHS INCHES).
    Now, ...  the below Calculation uses the SprialModel to CONFIRM the THICHNESS
     CALCULATION.
  ================================================================================
 *)

Clear[L, R, c, t]

SpiralModel[L_, R_, c_, t_] := L - Integrate[Sqrt[1 + ((2 π)/t u)^2], {u, c, R}]

R = 2.0 ;
c = 1.5  ;
t = 0.002    ;
(* L = ?  *) ;

Solve[SpiralModel[L, R, c, t] == 0, L]

{{L → 2748.89361767709194`}}
(*  VOILA !!!  We see that the SprialModel CONFIRMS the THICKNESS CALCULATION
               determined by the AreaModel !!!                                *)
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$$\text{SpiralModel}[L\_, R\_, c\_, t\_] := L - \text{Integrate}\left[\sqrt{1+\left(\frac{2\pi}{t}u\right)^2},\ \{u, c, R\}\right]$$