Monic Adjuncts of Quadratics

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Article 00.: Introduction. Given a univariant, Quadratic polynomial with, say, real number coefficients,

(0.1) $\quad \mathrm{Q}(x) = ax^2 + bx + c\,,$

there is an associated monic, quadratic polynomial, $M(x),$ which is described by the following,

> DEFINITION: To each quadratic polynomial as in (0.1), there is an associated monic, quadratic polynomial, $M(x),$ called the *monic adjunct of* $Q(x),$ and which is defined by,

(0.2) $\quad M(x) \;=\; x^2 + 2bx + 4ac \quad$ (*monic adjunct of Q*) .

As a consequence of some casual, algebraic, gymnastic doodling, several functional relationships (and connections) regarding the *zeros* of $Q(x)$ and $M(x)$ did emerge. An account of such doodling--and the results there of--are below presented in the following sequel of Articles.

Article 01.: Monic (Quadratic) Adjunct Polynomials. Consider a quadratic, $Q(x),$ as in Item (0.1). Now, note that

(1.1) $\quad 4a\,Q(x) \;=\; 4a^2x^2 + 4abx \,+ 4ac \;=\; (2ax)^2 + 2b(2ax) \,+ 4ac\,.$

So by declaring that $z = 2ax;$ and, that $M(z)$ be defined as

(1.2) $\quad M(z) \;= z^2 + 2bz + 4ac \;\;,$

the following relationships emerge.

(1.3) $\quad \boxed{\text{(A): } 4a\,Q(x) \;= M(2ax) \quad \text{and} \quad \text{(B): } Q(\frac{z}{2a}) \;=\; \frac{1}{4a}\,M(z)}\;.$

Appealing to (1.2), further exploration in such manner reveals that (for $z \neq 0$)

(1.4) $\quad M(\frac{4ac}{z}) \;=\; \frac{16a^2c^2 + 2b\cdot 4acz + 4acz^2}{z^2} \;=\; \frac{4ac}{z^2}\,M(z)\,.$

And then, from applications of Items (1.4) and (1.3A), we have that

(1.5) $\quad \frac{4ac}{z^2}\,M(z) = M(\frac{4ac}{z}) \;= M(2a\cdot\frac{2c}{z}) = 4a\,Q(\frac{2c}{z})\,,$

from which we conclude that $(\text{for } z \neq 0)$

(1.6) $$Q(\frac{2c}{z}) = \frac{c}{z^2} M(z) .$$

Items (1.3B), (1.4) and (1.6) are now, hereby, organized together and conveniently displayed as a below boxed bundle of relationships in the spirit of convenience and ease of application.

(1.7) $$\boxed{\text{A: } Q(\frac{z}{2a}) = \frac{1}{4a} M(z) \,\Big\|\, \text{B: } M(\frac{4ac}{z}) = \frac{4ac}{z^2} M(z) \,\Big\|\, \text{C: } Q(\frac{2c}{z}) = \frac{c}{z^2} M(z)} .$$

Appealing to the relationships of (1.7), another interesting functional relationship presents itself. Hence, consider the following

Theorem 1. Given a quadratic Q and its *monic adjunct* M, suppose that that for $z, w \in \mathbb{C}: \ zw = 4ac \neq 0$. Then: $zM(w) = wM(z)$.

Proof: Referencing Item (1.7B), note that,
$M(w) = M(\frac{zw}{z}) = M(\frac{4ac}{z}) = \frac{4ac}{z^2} M(z) = \frac{zw}{z^2} M(z) = \frac{w}{z} M(z)$.
Therefore, it now follows that,

(1.8) $$\boxed{zM(w) = wM(z)} . \quad \square$$

Article 02.: Novel Curiosities of *Q and M* Pairs . By implementing the functional relationships between a quadratic Q and its *monic adjunct* M, novel relationships involving the *zeros* of such Q *and* M polynomial pairs can be developed. We begin with the following theorem demonstrations.

Theorem 2. Given a quadratic Q and its monic adjunct M, suppose that for $z, w \in \mathbb{C}: \ zw = 4ac \neq 0$. Then: $M(z) = 0 \ \Leftrightarrow \ M(w) = 0$.

Proof: Given the *hypothesis*, (1.8) is applicable. Also, $z \neq 0 \neq w$. Hence, by (1.8) : $wM(z) = 0 \ \Leftrightarrow \ zM(w) = 0$. Thus, we have that,
$M(z) = 0 \ \Leftrightarrow \ wM(z) = 0 \ \Leftrightarrow \ zM(w) = 0 \ \Leftrightarrow \ M(w) = 0$.
Therefore: $M(z) = 0 \ \Leftrightarrow \ M(w) = 0$. $\square$

COROLLARY 2.1 *If* $zw - 4ac = 0$ for two *non-zero values, z,w* $\in \mathbb{C}$, *then: Either both are zeros of* M, *or else, Neither is a zero of M.*

Zeros of a quadratic polynomial, Q, and those of its *monic adjunct,* M, can now be related to each other by applying--in concert with each other--the results of Item (1.7A) together with Theorem 2.

Theorem 3. Given a quadratic Q and its monic adjunct M, suppose that $M(z) = 0$ for $z \in \mathbb{C}$ where $z \neq 0$. Then: $Q(\frac{z}{2a}) = 0 = Q(\frac{2c}{z})$.

Proof: Given that $M(z) = 0$ and $z \neq 0$, and then appealing to Items (1.7A) and (1.7C), it follows that: $Q(\frac{z}{2a}) = \frac{1}{4a} M(z) = 0 = \frac{c}{z^2} M(z) = Q(\frac{2c}{z})$. ☐

COROLLARY. Given a quadratic (polynomial) $Q(x)$, suppose that $z \neq 0$, and that z is a *fixed zero* of the *monic adjunct* $M(x)$. Then, *zeros of* $Q(x)$ are given by the formulas

(2.1) $$\boxed{x_1 = \frac{z}{2a} \quad \text{and} \quad x_2 = \frac{2c}{z}\ , \quad (z \neq 0)}\ .$$

Theorem 4. Given a quadratic Q and its monic adjunct M, suppose that $M(z) = 0$ for $z \in \mathbb{C}$ where $z \neq 0$. Then, *the zeros of* $Q(x)$ given by the formulas in Item (2.1) are *distinct zeros iff* $z \neq -b$.

Proof: We proceed to demonstrate that: $x_1 = x_2 \Leftrightarrow z = -b$.
Hence, for $z \neq 0$ and $M(z) = 0$, suppose that $x_1 = x_2$. Then,
$x_1 = x_2 \Rightarrow \frac{z}{2a} = \frac{2c}{z} \Rightarrow z^2 - 4ac = 0$. Also, since $M(z) = 0$,
$z^2 + 2bz + 4ac = 0$. Adding equations in z, we arrive at
$0 = 2z^2 + 2bz = 2z(z+b)$. Given $z \neq 0$, it we conclude $z = -b$.
Consequently, the implication: $x_1 = x_2 \Rightarrow z = -b$ follows.
Conversely, now suppose that $z = -b$. Then, we see that
$0 = M(z) = M(-b) = (-b)^2 + 2b(-b) + 4ac = -b^2 + 4ac$.
Thus, $z = -b \Rightarrow -b^2 + 4ac = 0 \Rightarrow \frac{-b}{2a} = \frac{2c}{-b} \Rightarrow x_1 = x_2$.
Therefore, we have established: $x_1 = x_2 \Leftrightarrow z = -b$. Hence,
the inverse biconditional: $x_1 \neq x_2 \Leftrightarrow z \neq -b$ can be concluded. ☐

Theorem 5. Given a quadratic (polynomial) $Q(x)$, suppose that $z \neq 0$, and that z is a *fixed zero* of the *monic adjunct* $M(x)$. Then, *zeros of* $Q(x)$, x_1 and x_2, given by the formulas of Item (2.1) satisfy the following equalities:

(2.2) $$\boxed{\text{(i)}\ x_1 + x_2 = -\frac{b}{a} \quad \text{and} \quad \text{(ii)}\ x_1 \cdot x_2 = \frac{c}{a}}\ .$$

Proof: Given that $M(z) = 0$ and $z \neq 0$, and observing that
$0 = M(z) = z^2 + 2bz + 4ac \Rightarrow z^2 + 4ac = -2bz$.
Hence, $x_1 + x_2 = \frac{z}{2a} + \frac{2c}{z} = \frac{z^2+4ac}{2az} = \frac{-2bz}{2az} = -\frac{b}{a}$.
And, also, $x_1 \cdot x_2 = \frac{z}{2a} \cdot \frac{2c}{z} = \frac{c}{a}$. ☐

In order to present the developments (and observations) of the preceding Articles as a conveniently organized list of *itemized results,* a *summary of such results* is now displayed below.

<u>Summary of Established Results and Observations</u>:

(2.3) For $z \in \mathbb{C}$ and $z \neq 0$: z is a *zero of* M *iff* $\frac{z}{2a}$ is a *zero of* Q.

(2.4) For $z \in \mathbb{C}$ and $z \neq 0$: z is a *zero of* M *iff* $\frac{4ac}{z}$ is a *zero of* M.

(2.5) For $z \in \mathbb{C}$ and $z \neq 0$, suppose that z is a *zero of* M, then:

$$\boxed{(A)\text{: } \frac{z}{2a} \text{ and } \frac{2c}{z} \text{ are zeros of } Q \;\Big\|\; (B)\text{: Distinct zeros iff } z \neq -b}.$$

(2.6) For $z \in \mathbb{C}$ and $z \neq 0$, suppose that z is a *zero of* M*;* then, the *zeros of* Q, x_1 *and* x_2 , displayed in Item (2.1) satisfy the equalities

$$\boxed{(i)\;\; x_1 + x_2 \;=\; -\frac{b}{a} \quad \text{and} \quad (ii)\;\; x_1 \cdot x_2 \;=\; \frac{c}{a}}.$$

(2.7) Note that the condition, $4ac \neq 0$, *restricts* Q from having zero values for *either* the *leading coefficient, a, or* the *constant term, c.* Of course, the condition that $a \neq 0$ is *implicit;* otherwise, Q would *not be a quadratic.* Observe that $c \neq 0$ excludes the instance of Q with a *zero* which has a *zero value.*

(2.8) The condition that $b^2 + c^2 \neq 0$ would restrict the *admittance* of the *trivial* Q, $ax^2 = 0$. Hence, exclusion of such trivial Q can be assured by considering only *admissible quadratic (polynomials)* Q as defined by the following

(2.9) <u>DEFINITION</u>: Degenerate and trivial quadratics, Q, can be excluded by considering only *admissible quadratics,* Q, which satisfy the condition that: $a \cdot (b^2 + c^2) \neq 0$.

<u>Article 03.: Alternative Renditions to the Quadratic Formula</u>. By appealing to the results of the preceding *Articles, novel and alternative renditions* to the standard *quadratic formula* can be developed and formulated. In order to accomplish this objective, we introduce the following *expression references.*

<u>DEFINITION</u>: Given a general quadratic equation, $Q(x) = 0$, the expressions created by the *plus and minus sign options* in the formula,

(3.1) $$\boxed{E \;=\; b \pm \sqrt{b^2 - 4ac}}$$, *(root effectors formula)*

are hereby defined as *quadratic root Effectors*.

(3.2) We now note the following observations regarding *quadratic root effectors:*

(i) *Degenerate and trivial quadratics* are avoided by considering only *admissible forms* which satisfy the condition: $a \cdot (b^2 + c^2) \neq 0$ (as detailed in Item (2.9)).

(ii) $E \neq 0$, for <u>*at least one*</u> of the *root effector* options (for *admissible forms*).

<u>Theorem 6</u>. The roots of an *admissible* quadratic equation, $Q(x) = 0$, can be determined by using the following *quadratic root effector formulas* $(QREF)$ with a *non-zero root effector.*

(3.3) $$\boxed{x_1 = -\frac{E}{2a} \quad \text{and} \quad x_2 = -\frac{2c}{E}} \;. \; (QREF)$$

<u>Proof</u>: From (3.2) we see that there does exist *at least one non-zero root effector* option of that *admissible Q.* Further, we note that for the *monic adjunct,* M, we have that $M(x) = x^2 + 2bx + b^2 - (b^2 - 4ac) = (x + b)^2 - (b^2 - 4ac)$, and, consequently, $M(-E) = [(-b \pm \sqrt{b^2 - 4ac}\,) + b]^2 - (b^2 - 4ac) = 0$, for *either option* of the *root effectors.* Now, by applying an $E \neq 0$ option, $(-E) \neq 0$ and $M(-E) = 0$, together with (2.1) of Theorem 3, establishes Item (3.3) , and so, completes this proof. ☐

<u>Theorem 7</u>. The roots of an *admissible* quadratic equation, $Q(x) = 0$, which are formulated by the *root effector formulas* of Item (3.3) satisfy the equalities

(3.4) $$\boxed{\text{(i) } x_1 + x_2 = -\frac{b}{a} \quad \text{and} \quad \text{(ii) } x_1 \cdot x_2 = \frac{c}{a}}\,.$$

Proof: Here, we simply apply Theorem 5 where: $z = (-E) \neq 0$ and observing that $M(-E) = 0$ as developed in the *proof* of Theorem 6. ☐

Having observed the fundamental relationships and connective interplay between a *quadratic* Q and its *monic adjunct M* as presented in the preceding Articles, this next result should actually come a as no surprise.

<u>Theorem 8</u>. The roots of an *admissible* quadratic equation, $Q(x) = 0$, as determined by *quadratic root effector formulas* of Item (3.3), can be classified by the *monic adjunct discriminant* $M(-b)$ *as follows:*

(3.5)

(i) $M(-b) < 0$	$\Rightarrow$	*distinct, real roots*	
(ii) $M(-b) = 0$	$\Rightarrow$	*one, real root*	
(iii) $M(-b) > 0$	$\Rightarrow$	*complex conjugate roots*	

Proof: Note that $M(-b) = (-b)^2 + 2b(-b) + 4ac = -b^2 + 4ac$. Now, by referencing the *root effectors formula* in Item (3.1) and observing that since $-M(-b)$ is, in fact, the *radicand* thereof, then the *parity and value* of $M(-b)$ clearly *classify the roots* of $Q(x) = 0$ as described in Item (3.5). ☐

EXAMPLE #01 REGARDING QUADRATIC ADJUNCTS

```
(*  Now consider the Given Quadratic:  Q (x) = 3 x^2-5 x+2.  Then,
    M (x) = x^2-10 x+24. Now proceed with the solution process.
    The symbol "E" is Protected. Hence, we use "R" to designate E  *)

Clear[a, b, c, R, x1, x2]
Q[x_] := a x^2 + b x + c
M[x_] := x^2 - 10 x + 24
a = 3 ; b = -5 ; c = 2 ;
SequenceForm[ "M(-b) = " , M[-b], " < 0 " ]
SequenceForm[ "R ∈ " , { b + Sqrt[b^2 - 4 a c] , b - Sqrt[b^2 - 4 a c] } ]
```

M(-b) = -1 < 0

R ∈ {-4, -6}

```
M(-b) = -1 < 0  (* Monic Discriminant < 0 ==> Two Real Zeros *)
```

```
(* Select R = -4, and then calculate x1 and x2              *)
R = -4 ;
SequenceForm[ "x1 = " , (-R)/(2 a) , "  and  " , "x2 = " , (2 c)/(-R) ]
```

$x1 = \frac{2}{3}$ and $x2 = 1$

```
(*  Checking these x-values, ... we find that               *)
SequenceForm[ "Q(x1) = ", Q[2/3] , "  and  " , "Q(x2) = ", Q[1] ]
```

Q(x1) = 0 and Q(x2) = 0

```
Q(x1) = 0  and  Q(x2) = 0  (* Illustration_#01 is COMPLETE  *)

(* ================================================================= *)
```

EXAMPLE #02 ILLUSTRATIVE DETAILS PRESENTED ON NEXT PAGE

EXAMPLE #02 REGARDING QUADRATIC ADJUNCTS

```
(*  Now suppose that:Q (x) = 9 x^2-12 x+5.  Then, M (x) = x^2-24 x+180.
    Proceeding with symbol "R" to designate E, we have that ...      *)

Clear[a, b, c, R, x1, x2]
Q[x_] := a x^2 + b x + c
M[x_] := x^2 + 2 b x + 4 a c
a = 9 ; b = -12 ; c = 5 ;
SequenceForm[ "M (-b) = " , M[-b], " > 0 " ]
SequenceForm[ "R ∈ " , { b + Sqrt[b^2 - 4 a c] , b - Sqrt[b^2 - 4 a c] } ]
```

```
M(-b) = 36 > 0
```

```
M(-b) = 36 > 0  (* Monic Discriminant > 0 ==> Complex Zeros      *)
```

```
R ∈ {-12 + 6 I, -12 - 6 I}
```

```
(* Select R = -12+6 I, and then calculate x1 and x2              *)
R = -12 + 6 I ;
```

```
SequenceForm[ "x1 = " , (-R)/(2 a) , "  and  " , "x2 = " , (2 c)/(-R) ]
```

$$x1 = \frac{2}{3} - \frac{I}{3} \quad \text{and} \quad x2 = \frac{2}{3} + \frac{I}{3}$$

```
(*  Checking these x-values, ... we find that                    *)

SequenceForm[ "Q(x1) = ", Q[2/3 - I/3] , "  and  " , "Q(x2) = ", Q[2/3 + I/3] ]
```

```
Q(x1) = 0  and  Q(x2) = 0
```

```
Q(x1) = 0  and  Q(x2) = 0  (* Illustration_#01 is COMPLETE   *)
```

EXAMPLE #03 REGARDING QUADRATIC ADJUNCTS

```
(*  Imagine a Quadratic Polynomial Q (x) over the Finite Field of integers
   Modulo (13), GF (13), and denoted by the symbol Z13 .  This illustration
   computes the ZEROS of the Given Quadratic,  Q (x) = 12 x^2+5 x+9, over Z13,
   by implementing the methods of this composition.
======================================================================
  In the spirit of convenience,... both: (SqrList)-- the Squares of Z13  and,
  also, (InvList)-- the Multiplicative Inverses of Z13, are hereby listed
  in order of: SqrList = {1^2 through to 13^2}; AND InvList = {1^-1 through to 13^-1} ,
  respectively.  Now, ...   consider the following results from Mathematica.
  Note: The Symbol "E" is PROTECTED in Mathematica; here, the Symbol "R" will
  INSTEAD be used AS A SUBSTITUTE for the SYMBOL "E" in what follows here.   *)

SqrList = PowerMod[{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12} , 2 , 13];
InvList = PowerMod[{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12} , -1 , 13];
SequenceForm["SqrList = " ,  SqrList ]
SequenceForm[ "InvList = " ,  InvList ]
(*  And, so ...                                                          *)
```

```
SqrList = {1, 4, 9, 3, 12, 10, 10, 12, 3, 9, 4, 1}
```

```
InvList = {1, 7, 9, 10, 8, 11, 2, 5, 3, 4, 6, 12}
```

DETAILS & DEVELOPMENT CONTINUED ON NEXT PAGE

```
(*  Applying the Results of this Composition to Q (x), below, note that      *)

Q[x_] := 12 x^2 + 5 x + 9
Clear[a, b, c, R, x1, x2]
a = 12; b = 5; c = 9;
SequenceForm["R ∈ ", {b + √Mod[b² - 4 a c, 13] ,  b - √Mod[b² - 4 a c, 13] }]

(*  Hence, we observe that                                                     *)
R ∈ {8, 2}
(* Therefore, applying the Root Effector Formulas by selecting R = 8 ,        *)
R = 8;
x1 = Mod[(-R)/(2 a) , 13]; x2 = Mod[(2 c)/(-R) , 13] ;
SequenceForm["x1 = ", x1 , "  and  " , "x2 = ", x2 ]
(*  And so, we have that ...                                                  *)
```

$x1 = \frac{38}{3}$ and $x2 = \frac{43}{4}$

```
x1 = 38/3  and  x2 = 43/4  (* So that Mod (13), x1 = 38*3^-1  and  x2 = 43*4^-1    *)
```

```
Clear[x1, x2]
x1 = Mod[38 * 9, 13]; x2 = Mod[43 * 10 , 13] ; (* Appealing the "InvList" above  *)
SequenceForm[ "x1 = ",  x1 , "  and  " , "x2 = ", x2 ]
```

```
(* Consequently it follows that .........................................*)
x1 = 4  and  x2 = 1
```

```
SequenceForm[ "Q(4) = ", Mod[Q[4] , 13], "  and  " , "Q(1) = " , Mod[Q[1], 13] ]
```

```
(*   These x-values ARE ZEROS via the above Mathematica Code   ...........*)
Q(4) = 0  and  Q(1) = 0
(* Hence, THIS EXAMPLE #03 IS COMPLETE ....................................... *)
```