(Simple) BROKEN-LINE Diagonals of Rectangles

(Developed, Composed & Typeset by: J B Barksdale Jr / 05-26-15)

Article 00.: Introduction. Imagine the image of a rectangle (say, in $\mathbb{R}^2$) with the diagonals drawn as *(simple) broken-lines which SHARE a SINGLE, COMMON break-point, say D (see Figure_01, below).* Now, suppose the expression $(e+f)$ denotes the *sum of the segment lengths* of the first *broken-line diagonal*; and that the expression $(g+h)$ denotes the *sum of the segment lengths* of the other *broken-line diagonal*. The purpose of this developmental excursion is to demonstrate a rather *curious and novel* attribute of such *broken-line diagonals;* namely, regarding such *broken-line diagonal segments*, it always follows that

(0.1) $\quad e^2 + f^2 \; = \; g^2 + h^2$, (for *arbitrary* *Rectangle $\mathcal{R}$* and *arbitrary Point D*) .

From the above description, one would imagine the *break-point, D,* through which the *broken-line diagonals pass,* to be an *interior point* of the *rectangle.* However, the following developments establish that the *point, D,* can actually be an *arbitrary point of* $\mathbb{R}^2$ (regarding the above, given details); hence, the *point, D,* can be an *interior point, exterior point, or edge point* of the given rectangle.

Article 01.: Equality & Sums of Squared Segments. *Figure_01*, below, visually illustrates the descriptions of an *arbitrary rectangle* (in $\mathbb{R}^2$), and *(simple) broken-line diagonals* which pass through a single *arbitrary break-point, D,* as presented in *Article_00*, above.

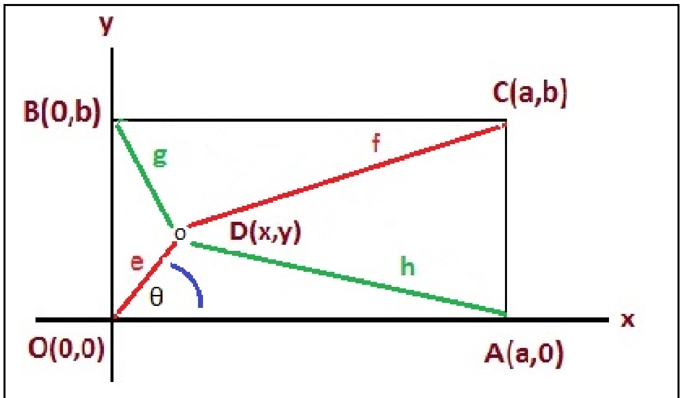


Figure_01: Rectangle with *broken-line diagonals*

By appealing to the *distance formula (for $\mathbb{R}^2$)*, the following formulations regarding the *broken-line diagonal segment lengths* are clearly rendered.

(1.1) $\quad e^2 \; = \; x^2 + y^2$ and $f^2 \; = \; (x-a)^2 + (y-b)^2$.

Also,

(1.2) $\quad g^2 \; = \; x^2 + (y-b)^2$ and $h^2 \; = \; (x-a)^2 + y^2$.

Now, by adding the pairs of equations in each of Items (1.1) and (1.2), it follows that

(1.3) $\quad e^2 + f^2 \;=\; [x^2 + (y-b)^2] + [y^2 + (x-a)^2] \;=\; g^2 + h^2$.

The preceding developments establish the following

Theorem 1. Given a *rectangle,* $\mathcal{R} = \{\, (\kappa a, \lambda b) \,\big|\, \kappa, \lambda \in [0,1] \,\} \subset \mathbb{R}^2$ with *horizontal edges* $= a$ units and with *vertical edges* $= b$ units , suppose that *arbitrary* internal *broken-line diagonal-segment length pairs* have measures of e and f for a first broken-line diagonal, and value measures of g and h for the other broken-line diagonal. Then,

(1.4) $$\boxed{e^2 + f^2 \;=\; g^2 + h^2} \;.$$

Proof: The development of Items (1.1) – (1.3), and reference to *Figure_01,* above, establishes the conclusion as a consequence of the given hypothesis. ☐

COROLLARY (1-A). Item (1.4) is a consequence of the hypothesis of ***Theorem 1*** for *an arbitrary break-point,* $D \in \mathbb{R}^2$.

Proof: Although *Figure_01* depicts the *break-point* restriction, $D \in \mathcal{R}$; actually, the development of Items (1.1) – (1.3) remains unaltered for *an arbitrary point,* $D \in (\mathbb{R}^2 \backslash \mathcal{R})$ as well. Therefore, the conclusion of *Theorem 1* remains intact for *any point* $D \in \mathbb{R}^2$. ☐

Article 02.: Broken-line Diagonal Results via Vector Methods. A *vector diagram version* of *Figure-01* is now presented via *Figure_02,* below. Consider the rectangle $\mathcal{R} = \{\, (\kappa \vec{\mathbf{a}} + \lambda \vec{\mathbf{b}}\,) \,\big|\, \kappa, \lambda \in [0,1]\,,\, \vec{\mathbf{a}} \perp \vec{\mathbf{b}} \,\}$ and a given *arbitrary point D* which has *position vector* $\vec{\mathbf{e}}$. Now, with given *vector diagram references* $\vec{\mathbf{f}}$, $\vec{\mathbf{g}}$ and $\vec{\mathbf{h}}$, we define

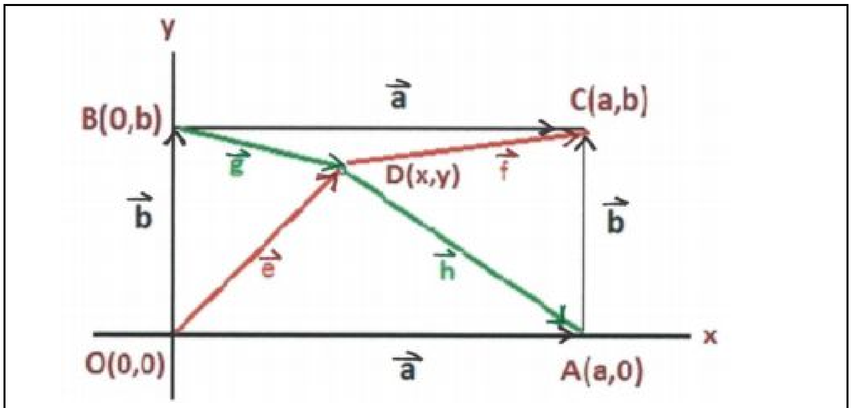


Figure_02: *Vector Diagram* with broken-line diagonal vectors

(2.1) $$\boxed{(i):\; \vec{\mathbf{f}} = \vec{\mathbf{a}} + \vec{\mathbf{b}} - \vec{\mathbf{e}} \qquad (ii):\; \vec{\mathbf{g}} = \vec{\mathbf{e}} - \vec{\mathbf{b}} \qquad (iii):\; \vec{\mathbf{h}} = \vec{\mathbf{a}} - \vec{\mathbf{e}}}$$

From $\vec{\mathbf{a}} \perp \vec{\mathbf{b}}$ (so that $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0$) and the equalities appearing in Item (2.1), we have

(2.2) $\quad \overrightarrow{\mathbf{g}} \cdot \overrightarrow{\mathbf{h}} = (\overrightarrow{\mathbf{e}} - \overrightarrow{\mathbf{b}}) \cdot (\overrightarrow{\mathbf{a}} - \overrightarrow{\mathbf{e}}) = \overrightarrow{\mathbf{e}} \cdot \overrightarrow{\mathbf{a}} - \overrightarrow{\mathbf{e}} \cdot \overrightarrow{\mathbf{e}} - \overrightarrow{\mathbf{b}} \cdot \overrightarrow{\mathbf{a}} + \overrightarrow{\mathbf{b}} \cdot \overrightarrow{\mathbf{e}}$
$\quad\quad = \overrightarrow{\mathbf{e}} \cdot (\overrightarrow{\mathbf{a}} + \overrightarrow{\mathbf{b}} - \overrightarrow{\mathbf{e}}) = \overrightarrow{\mathbf{e}} \cdot \overrightarrow{\mathbf{f}}$.

Thus, the *vector diagram (Figure_02)* together with Items (2.1) and (2.2) clearly establish

Theorem 2. Given a *rectangle* $\mathcal{R} = \{ (\kappa \overrightarrow{\mathbf{a}} + \lambda \overrightarrow{\mathbf{b}}) \,\big|\, \kappa, \lambda \in [0,1] \,, \overrightarrow{\mathbf{a}} \perp \overrightarrow{\mathbf{b}} \} \subset \mathbb{R}^2$ and *arbitrary point* $D \in \mathbb{R}^2$ with *position vector* $\overrightarrow{\mathbf{e}}$, then the *broken-line diagonal-segment vectors* $\overrightarrow{\mathbf{e}}$ and $\overrightarrow{\mathbf{f}}$, $\overrightarrow{\mathbf{g}}$, $\overrightarrow{\mathbf{h}}$ as defined in Item (2.1) satisfy the equality,

(2.3) $$\boxed{\overrightarrow{\mathbf{e}} \cdot \overrightarrow{\mathbf{f}} = \overrightarrow{\mathbf{g}} \cdot \overrightarrow{\mathbf{h}}} \;.$$

Further examination of Item (2.1) and the vector diagram in *Figure_02* effectively render

Theorem 3. Given the *hypothesis of Theorem 2*, Item (2.1) and the *vector diagram in Figure_02*, it follows that

(2.4) $$\boxed{\| \overrightarrow{\mathbf{e}} \|^2 + \| \overrightarrow{\mathbf{f}} \|^2 = \| \overrightarrow{\mathbf{g}} \|^2 + \| \overrightarrow{\mathbf{h}} \|^2 \iff \overrightarrow{\mathbf{e}} \cdot \overrightarrow{\mathbf{f}} = \overrightarrow{\mathbf{g}} \cdot \overrightarrow{\mathbf{h}}}$$

Proof: Suppose the *hypothesis;* now, inspect *Figure_02* , and Item (2.1), to conclude that

(2.5) $\quad \overrightarrow{\mathbf{e}} + \overrightarrow{\mathbf{f}} = \overrightarrow{\mathbf{a}} + \overrightarrow{\mathbf{b}}$ and $\overrightarrow{\mathbf{a}} - \overrightarrow{\mathbf{b}} = \overrightarrow{\mathbf{g}} + \overrightarrow{\mathbf{h}}$.

Since $\overrightarrow{\mathbf{a}} \cdot \overrightarrow{\mathbf{b}} = 0$, we can conclude that,

(2.6) $\quad \| \overrightarrow{\mathbf{a}} + \overrightarrow{\mathbf{b}} \|^2 = \| \overrightarrow{\mathbf{a}} \|^2 + \| \overrightarrow{\mathbf{b}} \|^2 + 2 (\overrightarrow{\mathbf{a}} \cdot \overrightarrow{\mathbf{b}})$
$\quad\quad = \| \overrightarrow{\mathbf{a}} \|^2 + \| \overrightarrow{\mathbf{b}} \|^2 - 2 (\overrightarrow{\mathbf{a}} \cdot \overrightarrow{\mathbf{b}}) = \| \overrightarrow{\mathbf{a}} - \overrightarrow{\mathbf{b}} \|^2$.

Now, from Items (2.5) and (2.6) we have

(2.7) $\quad \| \overrightarrow{\mathbf{e}} + \overrightarrow{\mathbf{f}} \|^2 = \| \overrightarrow{\mathbf{a}} + \overrightarrow{\mathbf{b}} \|^2 = \| \overrightarrow{\mathbf{a}} - \overrightarrow{\mathbf{b}} \|^2 = \| \overrightarrow{\mathbf{g}} + \overrightarrow{\mathbf{h}} \|^2$.

Item (2.7) now yields

(2.8) $\quad \| \overrightarrow{\mathbf{e}} + \overrightarrow{\mathbf{f}} \|^2 = \| \overrightarrow{\mathbf{e}} \|^2 + \| \overrightarrow{\mathbf{f}} \|^2 + 2 (\overrightarrow{\mathbf{e}} \cdot \overrightarrow{\mathbf{f}})$
$\quad\quad = \| \overrightarrow{\mathbf{g}} \|^2 + \| \overrightarrow{\mathbf{h}} \|^2 + 2 (\overrightarrow{\mathbf{g}} \cdot \overrightarrow{\mathbf{h}}) = \| \overrightarrow{\mathbf{g}} + \overrightarrow{\mathbf{h}} \|^2$.

Item (2.8) then renders

(2.9) $$\Big[(\| \overrightarrow{\mathbf{e}} \|^2 + \| \overrightarrow{\mathbf{f}} \|^2) - (\| \overrightarrow{\mathbf{g}} \|^2 + \| \overrightarrow{\mathbf{h}} \|^2) \Big] = (-2) \Big[(\overrightarrow{\mathbf{e}} \cdot \overrightarrow{\mathbf{f}}) - (\overrightarrow{\mathbf{g}} \cdot \overrightarrow{\mathbf{h}}) \Big].$$

The biconditional conclusion of Item (2.4) is now thus asserted by Item (2.9) ☐

Theorem 4. Given a *rectangle* $\mathcal{R} = \{ (\kappa \vec{\mathbf{a}} + \lambda \vec{\mathbf{b}}) \,\big|\, \kappa, \lambda \in [0,1], \vec{\mathbf{a}} \perp \vec{\mathbf{b}} \} \subset \mathbb{R}^2$ and *arbitrary point* $D \in \mathbb{R}^2$ with *position vector* $\vec{\mathbf{e}}$, then the *broken-line diagonal-segment vectors* $\vec{\mathbf{e}}$ and $\vec{\mathbf{f}}$, $\vec{\mathbf{g}}$, $\vec{\mathbf{h}}$ as defined in Item (2.1) satisfy the equality,

(2.10) $\boxed{\| \vec{\mathbf{e}} \|^2 + \| \vec{\mathbf{f}} \|^2 = \| \vec{\mathbf{g}} \|^2 + \| \vec{\mathbf{h}} \|^2}$

Proof: Suppose the *hypothesis* and then *apply Theorems 2 and 3 , above.* ☐

Article 03.: Extensions to Euclidean $\mathbb{R}^n$ Spaces. The above displayed vector method developments clearly assert that the above *Theorems & Results* rely only on the *vector definitions, relationships, and inner product properties.* In order to illustrate this declaration for, say $\mathbb{R}^3$, consider vectors specified as in *Figure_02*, and given by

(3.1) $\vec{\mathbf{a}} = (a,0,0)\,; \quad \vec{\mathbf{b}} = (0,b,0)\,; \quad \vec{\mathbf{e}} = (x,y,z)\,.$

Then, by applying the *definitions* of $\vec{\mathbf{f}}$, $\vec{\mathbf{g}}$, $\vec{\mathbf{h}}$ as displayed in Item (2.1), it follows that

(3.2) $\vec{\mathbf{f}} = (a-x,\ b-y,\ -z)\ ; \quad \vec{\mathbf{g}} = (x,\ y-b,\ z)\,;$ and $\vec{\mathbf{h}} = (a-x,\ -y,\ -z)\,.$

Appealing to the *notations and inner product definitions* for $\mathbb{R}^3$, *Items (2.3) and (2.10)* can be established by *direct calculation.* Hence, by applying these *vector presentations for* $\mathbb{R}^3$,

(3.3) $\vec{\mathbf{e}} \cdot \vec{\mathbf{f}} = (x,y,z) \cdot (a-x,\ b-y,\ -z) = ax - x^2 + by - y^2 - z^2$

$$= ax - x^2 - y^2 + by - z^2 = (x, y-b, z) \cdot (a-x, -y, -z) = \vec{\mathbf{g}} \cdot \vec{\mathbf{h}}\,.$$

Also, note that,

(3.4) $\| \vec{\mathbf{e}} \|^2 + \| \vec{\mathbf{f}} \|^2 = [x^2+y^2+z^2] + [(a-x)^2+(b-y)^2+z^2]$

$$= [x^2+(y-b)^2+z^2] + [(a-x)^2+y^2+z^2] = \| \vec{\mathbf{g}} \|^2 + \| \vec{\mathbf{h}} \|^2\,.$$

Reviewing this present development, observe that the *vector references* reside in $\mathbb{R}^3$; thus, the *line segment (vector shafts)* constitute the <u>rectangle edges</u> and the <u>polyhedral edges</u> connecting the *given point, D,* to the <u>rectangle's vertices</u>. Hence, imagine *point D of Figure_01* as a point in $\mathbb{R}^3$ *which is elevated* out of the *xy-plane* by having, say, a *positive z-coordinate.* Then, the opposite, broken-line diagonal *red segments,* **e** *and* **f** , and the opposite, broken-line *green segments,* **g** *and* **h** , of *Figure_01* are actually *opposite (non-adjacent) polyhedral edges.* Curiously, however, Item (3.4), again, establishes that the lengths of such edges satisfy the equality therein presented.

Article 04.: Generalizations to Inner Product Spaces. Although extensions of the preceding developments to an arbitrary inner product space are *somewhat artificial* without the *spatial notions* of the supporting geometry, the preceding theorem results do remain intact by supplying the appropriate *vector notions and relationships* to replace that geometric support. Hence, let $(\mathbb{V}, \odot)$ denote an *inner product space*. Note that *Figure_02* of a *preceding Article* illustrates the following *geometric aspects* of that displayed rectangle for the *given orthogonal vectors* $\overrightarrow{\mathbf{a}}$ *and* $\overrightarrow{\mathbf{b}}$,

(4.1) *(i)* *vectors* $\overrightarrow{\mathbf{a}}$ *and* $\overrightarrow{\mathbf{b}}$ are the *edge vectors* of the *rectangle* $\mathcal{R}$.

(ii) $(\overrightarrow{\mathbf{a}} + \overrightarrow{\mathbf{b}})$ and $(\overrightarrow{\mathbf{a}} - \overrightarrow{\mathbf{b}})$ are the *diagonal vectors* of the *rectangle* $\mathcal{R}$.

(iii) *vector* $\overrightarrow{\mathbf{e}}$ is the *position vector* of the *given arbitrary break-point, D.*

(iv) *vectors* $\overrightarrow{\mathbf{e}}$, $\overrightarrow{\mathbf{f}}$, $\overrightarrow{\mathbf{g}}$ *and* $\overrightarrow{\mathbf{h}}$ as defined by *Item (2.1)* illustrate the *broken-line diagonal-segment vectors* in the *vector diagram* so that:

$$\overrightarrow{\mathbf{e}} + \overrightarrow{\mathbf{f}} = \overrightarrow{\mathbf{a}} + \overrightarrow{\mathbf{b}} \ (\textit{diag from } \mathbf{0}) \text{ and } \overrightarrow{\mathbf{g}} + \overrightarrow{\mathbf{h}} = \overrightarrow{\mathbf{a}} - \overrightarrow{\mathbf{b}} \ (\textit{diag from } \mathbf{B}).$$

Observe that by *defining and implementing* the *notions and relationships* of the expressions described in *Items (2.1) and (4.1)*, the state of all developments displayed in *Article_02* remain complete, unaffected and in force. This enforced state follows, of course, because the referenced vectors, definitions and relationships are *preserved and unchanged* in the developmental details of *Article_02*. Hence, by appealing to *notions and relationships* here discussed, we see a rather abstracted formulation of the above results which is stated below and presented as

Theorem 5. Given a *rectangle* $\mathcal{R} = \{ (\kappa\overrightarrow{\mathbf{a}} + \lambda\overrightarrow{\mathbf{b}}) \mid \kappa, \lambda \in [0,1], \overrightarrow{\mathbf{a}} \perp \overrightarrow{\mathbf{b}} \} \subset \mathbb{V}$, an *arbitrary-point vector* $\overrightarrow{\mathbf{e}} \in \mathbb{V}$, and *broken-line diagonal-segment vectors* $\overrightarrow{\mathbf{e}}$, $\overrightarrow{\mathbf{f}}$, $\overrightarrow{\mathbf{g}}$, *and* $\overrightarrow{\mathbf{h}}$ as defined in Item (2.1), it follows that

(4.2) $$\boxed{\| \overrightarrow{\mathbf{e}} \|^2 + \| \overrightarrow{\mathbf{f}} \|^2 = \| \overrightarrow{\mathbf{g}} \|^2 + \| \overrightarrow{\mathbf{h}} \|^2}$$

Proof: Suppose the *hypothesis;* now, simply appeal to the *given definitions and relationships* of the *vectors*, and the implementation of the *inner product* $(\odot)$ *properties;* then, *mimic* the developmental details appearing in *Article_02.* ☐

Article 05.: Metric Formulations. In the presence of *Figure_03 (next page)*, metric relationships and formulations among the *broken-line diagonal segment lengths*, the rectangle *edge lengths*, the *angle* θ, and the *coordinate values of the break-point D*, as modeled in *Figure_03*, can be formulated. In the preceding *Articles*, it was established that given an *arbitrary rectangle* and *an artitrary point*, $D(x, y)$, within such rectangle, the *Broken-Line Diagonal segment lengths:* e, f, g *and* h satisfy the *BLD equality:* $e^2 + f^2 = g^2 + h^2$.

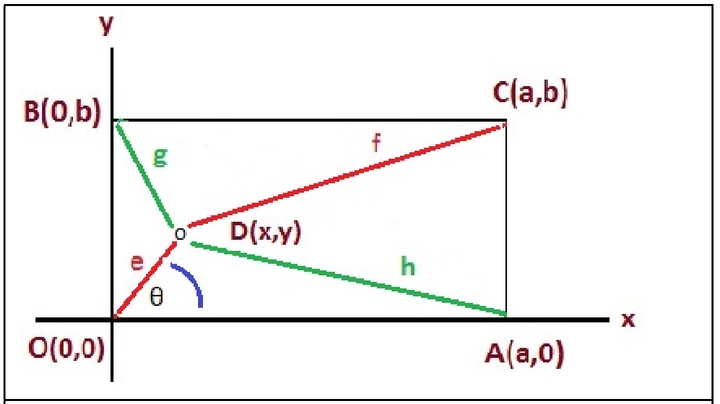


Figure_03: Rectangle with Broken-Line Diagonals

Clearly, given the presence of the *BLD equality,* only three of the *BLD segments* need be given in order to determine the fourth such segment. Hence, we proceed to investigate the modeling developments resulting from being given, say: e, g and h. In the spirit of convenience and simplicity of formulation, one vertex of the *given, aribitrary rectangle is placed at the Origin.* Also, this *modeling development* locates the *shortest BLD segment* at the origin. Further, this *modeling* supposes that the *BLD break-point lies inside* the rectangle. With these modeling details specified, proceed to *join BLD segments:* g and h to the *Break-Point, D;* then, *rotate:* h and g about *point D* so that *segment* h contacts the x*-axis*, and *segment* g contacts the y*-axis.* Label those *contact points:* $A(a,0)$ *and* $B(0,b)$. The *resulting point* $C(a,b)$ completes the fourth vertex of a rectangle whose *broken-line diagonal segments:* e, f, g, h satisfy the *BLD equality.*

In order to establish metric relationships among: e, g, h, a, b, x, y and θ; from *Figure_03,* observe that,

$$(5.11) \quad \boxed{\text{(A): } a \;=\; x + (a - x) \;=\; x + \sqrt{h^2 - y^2} \;\Big\|\; \text{(B): } b \;=\; y + (b - y) \;=\; y + \sqrt{g^2 - x^2}} \;.$$

Note that alternative presentations of the equalities in Item (5.11) can be formualted by inspecting: $(a-x)^2 = h^2 - y^2$ and $(b-y)^2 = g^2 - x^2$; now, recalling *this model* declares $x^2 + y^2 = e^2$, it follow that,

$$(5.12) \quad \boxed{\text{(X): } x \;=\; \frac{a^2 + e^2 - h^2}{2a} \quad \text{and} \quad \text{(Y): } y \;=\; \frac{b^2 + e^2 - g^2}{2b}} \;.$$

By viewing *Figure_03*, observe that,

$$(5.2) \quad \boxed{\text{(A): } x = e \cdot \cos\theta \quad \text{and} \quad \text{(B): } y = e \cdot \sin\theta} \;.$$

Then, applying Item (5.2) to the equations displayed in Item (5.11), it follows that,

$$(5.3) \quad \boxed{\text{(A): } a(\theta) = e \cdot \cos\theta + \sqrt{h^2 - e^2 \sin^2\theta} \;\Big\|\; \text{(B): } b(\theta) \;=\; e \cdot \sin\theta \;+\; \sqrt{g^2 - e^2 \cos^2\theta}} \;.$$

Since this model supposes that the *point* D *lies inside* the rectangle, then the *e-radius angle* θ lies in the *first quadrant;* hence, $\theta \in [0, \frac{\pi}{2}]$. Observe that the *e-radius* is *hinge-linked* to the *BLD-segments*, g *and* h. Also, by the *model description*, g *and* h remain in contact with their respective axes; thus, as the *e-radius* pivots *counter-clockwise* from $\theta = 0$ *to* $\theta = \frac{\pi}{2}$, the *points* A *and* B *track the axes* and (consequently) *graphically illustrate* all of the *BLD rectangles* which satisfy the *BLD equality:* $e^2 + f^2 = g^2 + h^2$.

By implementing the formulated relationships of the *modeling framework* presented in Items (5.11), the dimensions: *width* $= a$ and *height* $= b$, of an *accommodating rectangle* can be determined from the given *metrics data:* e, g and h, with x and y such that: $x^2 + y^2 = e^2$. Hence, for each *first quadrant point* $D(x, y)$ on the *circle:* $x^2 + y^2 = e^2$, there *exist a rectangle* satisfying the *BLD equality* for the given *data metric values:* e, g and h. Note that the *value f* is specified by the *BLD equality*: $f^2 = g^2 + h^2 - e^2$.

Alternatively, by *declaring a* θ*-value,* the *coordinates of an interior point* $D(x, y)$ and the *accommodating rectangle dimensions* are both rendered by applying Items (5.2) and (5.3).

Article 06.: BLD-Quadratures. This *Article_06* is devoted to exploring the notion of deciding the *existence* of a *Square*, and/or the *edge length* of a *Square* , and/or *declaring break-point coordinates* associated with a *Square* which satisfies the *BLD equality* for given *broken-line diagonal segment lengths: e, g* and h. A *Square* achieved by *contiuously altering* the *dimensions of an accommodating rectangle (by increasing the* θ*-angle)* until it becomes an *accommodating Square* is hereby declared to be a *BLD-Quadrature (Broken-Line Diagonal Quadrature).* (Note: Here, *Quadrature* DOES NOT refer to an integral nor integration method).

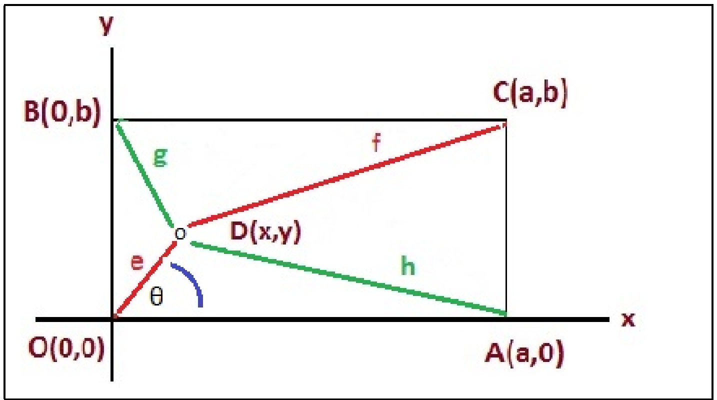


Figure_03: Rectangle with *Broken-Line Diagonals*

Inspecting *Figure_03* and Item (5.3), it is noted that $a(\theta)$ *decreases* and $b(\theta)$ *increases* for *increasing* $\theta \in [0, \frac{\pi}{2}]$. Hence,

(6.1) $\quad z(\theta) = b(\theta) - a(\theta)$,

is a *strictly increasing* function over $[0, \frac{\pi}{2}]$. Application of the *Intermediate Value Theorem* appears to render the following conclusions:

(6.2) *(i)* $\quad z(0) > 0$ *or* $z(\frac{\pi}{2}) < 0 \Rightarrow$ *NO BLD-Quadratures exist.*

(ii) $\quad z(0) < 0$ *and* $z(\frac{\pi}{2}) > 0 \Rightarrow$ *a unique BLD-Quadrature does exist.*

Application of the formulations in Item (5.3) render, (6.3) *(i)* $z(0) = \sqrt{g^2 - e^2} - (e+h)$ and *(ii)* $z(\frac{\pi}{2}) = (e+g) - \sqrt{h^2 - e^2}$.

From Item (6.3), it now follows that

(6.4) *(i)* $z(0) < (g - h - e)$ and *(ii)* $z(\frac{\pi}{2}) > (g - h + e)$.

Thus, if the *relative values* for the *BLD-segments:* $e,\ g, h$ are such that

(6.5) *(i)* $(g - h - e) < 0$ and *(ii)* $(g - h + e) > 0$,

then, it appears that: *a unique BLD-Quadrature does exist for that family of BLD rectangles.*
Similary: $[e^2 + (e{+}h)^2 < g^2 \ or\ h^2 > e^2 + (e{+}g)^2] \ \Rightarrow$ *NO BLD-Quadratures exist*.
A *Vertex Graph Plot* which includes a *Quadrature* occurrence is presented on the next page.

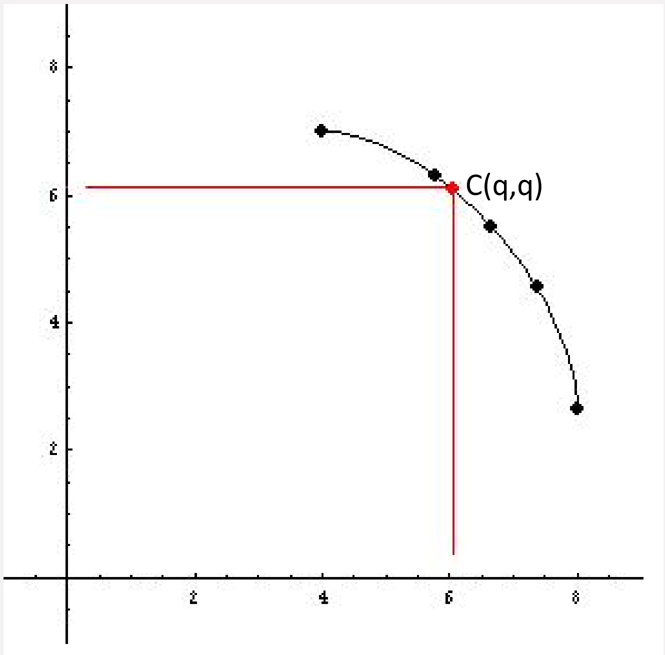
Article 07.: Computational Illustrations. In the presence of *Figure_03*, imagine that the *e-radius rotates* so that the *θ-angle increases from 0 to* $\frac{\pi}{2}$. Given numerical values for: $e,\ g$ *and* $h,$ an *animated illustration* of the family of *BLD rectangles* thus created by such rotating motion can be *mentally visualized.* By appealing to the formulations presented in the preceding articles of this composition, all of the *metrics* of such family members can be numerically computed. The *computational examples* regarding this composition are hereto attached among the last pages of this composition.

Computational Example: 01. Given data: $e = 25;\ g = 39;\ h = 52.$
For these data values, there are *infinitely many BLD rectangles.* However,
for a *specified point D,* there will exist a *unique BLD rectangle* for this data.
So, ... *Suppose* θ *is GIVEN by declaring that:* cos $\theta = .8000$.
This example presents the computational details to determine: *(i) point D*(x, y)*;*
and *(ii)* $a(\theta) =$ *rectangle width &* $b(\theta) =$ *rectangle height.*

Computational Example: 02. Given data: $e = 25;\ g = 39;\ h = 52;$
also, ... the *BLD rectangle* has a *Given Width of:* $a = 70.$
DETERMINE: *(i) Does such BLD rectangle actually exist?*
(ii) If so, ... Compute *point* $D(x, y);$ *(iii) If so, ...* Compute *Height* $= b.$

Computational Example: 03. Given data: $e = 25;\ g = 39;\ h = 52.$
(i) Does this data *support the Existence* of a *BLD Quadrature?*
(ii) If so, ... DETERMINE: the *Edge Length* $= q$ of such *BLD Quadrature* .

Article 08.: Concluding Remarks. In the presence of the *geometric visualization* of a rectangle as imagined in $\mathbb{R}^2$, or $\mathbb{R}^3$, and an imagined *arbitrary internal or external given point, D, Theorem 5* harbors a somewhat *curious, mystic* and *novel* tone. However, when stated in terms of a general *inner product space* $(\mathbb{V},\ \odot\ ),$ then the conclusion is simply a result from the exercise of *implementing vector definitions* and the *properties of an inner product,* $\odot$. Hence, the generalization does not appear to have any *hidden surprises* and/or *curious aspects*.

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(* Vertex Plots of Point C (a,b) as the e-radius
   rotates and increases from: t = 0  to t = π/2.
   Note that a BLD Quadrature has a Vertex at
   at Point C for:  a = 6.0715 = b .  Hence,
   this Data Set, the BLD Quadrature has
   Edge Length = q = 6.0715
```

```
Clear[a, b, z, v]
a[t_] := e Cos[t] + Sqrt[h^2 - e^2 Sin[t]^2]
b[t_] := e Sin[t] + Sqrt[g^2 - e^2 Cos[t]^2]
z[t_] := b[t] - a[t]

e = 3  ;
h = 5  ;
g = 4  ;
Solve[z[t] == 0, t] // N
{{t → 0.960994178657803743`}, {t → -2.90477822725276126`}}
v[t_] = {a[t], b[t]}
v[0.960994]
{6.07149698285807382`, 6.07149585414752657`}
```

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In this example, the variable "t" is used instead of the symbol "theta." Note that from the Given Data: 8 = e + h (is the largest value of a *BLD rectangle Width* at t = 0; and,
7 = e + g (is the largest value of a *BLD rectangle Height* at t = Pi/2 .

The <u>*BLD Quadrature Vertex*</u> for this Data Set occurs at: *t = 0.961* ; (angle degree-measure of about 55 degrees).

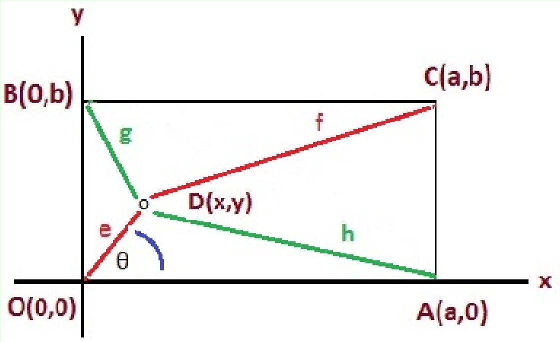
Referencing the above *Formulations* and *Computations,* note that:

(i) the *vertex function v[t]* calculates coordinates of Point C(a,b);
(ii) *z[t*] = 0* declares: *Rectangle Width = a[t*] = b[t*] = Rectangle Height ;*
(iii) Hence, the <u>*BLD Quadrature Edge Length*</u> *= q ;* where: a[t*] = q = b[t*] .

```
(*   This COMPUTATIONAL DEVELOPMENT illustrates how to compute
      the: WIDTH = a ; HEIGHT = b ; AND  POINT D (x,y) of a
      BLD-rectangle from GIVEN DATA which includes:
      e, g, h and the e-radius angle = θ .
    ========================================================== *)

(*   Given:  e, g, h  and specify θ  by:  cosθ = 0.8000

Clear[aA, bB, a, b, e, g, h, x, y, cosθ, sinθ]
e = 25  ;   (*  e = OD  by declaration   *)
g = 39  ;
h = 52  ;
f = √(g² + h² - e²) ; (* The FOURTH BLD-segment from D to C .
```




```
(* Calculate a (θ)   and  b (θ)  from:  cosθ = 0.8000 ;
   Hence:  sinθ = 0.6000 = √(1-(0.8)^2)  .  Formulate
   values for  x and y to compute Width = a and  Height = b  *)

Clear[aA, bB, x, y, cosθ, sinθ]
cosθ = 0.8000 ;
sinθ = 0.6000 ;
x = e cosθ ;
y = e sinθ ;
aA[h_, g_, x_, y_] := x + √(h² - y²)  ;
bB[h_, g_, x_, y_] := y + √(g² - x²)  ;

(*  Compute values for the pair: { aA , bB }              *)
a = aA[h, g, x, y] ;
b = bB[h, g, x, y] ;
{ a , b } // N

{69.7896 , 48.4813 }

(*   Hence:  a = Width = 69.7896  and  b = Height = 48.4813
     Now, List coordinates  x and y  for Point D (x,y) .     *)
{x, y}

{20. , 15. }

(*  So, ... Point D has coordinates: D (20 , 15)            *)


(* Now, ... Display Computed Values for: { a, b, x , y, f }. *)

{a, b, x, y, f} // N

{69.7896 , 48.4813 , 20. , 15. , 60. }
```

```
(*   Distance Formula Verification for: g, h and f            *)
```

$$\left\{g-\sqrt{(b-y)^2+x^2}\ ,\ h-\sqrt{(a-x)^2+y^2}\ ,\ f-\sqrt{(b-y)^2+(a-x)^2}\right\}\ //\ N$$

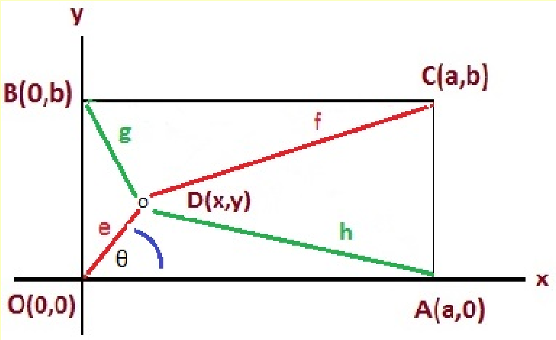
```
{0., 0., 0.}
```

```
(*  Values   {0,0,0}  confirm the Formulated Descriptions for
    { x , y , b }  do create the correct Calculated Values
    which do compute the correct measures for the
    BLD-segments: g, h, e, and f and for the Height = b       *)
```

```
(*   This COMPUTATIONAL DEVELOPMENT illustrates how to compute
      the HEIGHT of a BLD-rectangle from GIVEN DATA which
      includes:  e, g, h and the Width-Value = a .
    =============================================================   *)

(*   Given:  e, g, h   and  declared  width-value = a

Clear[bB, xD, yD, a, b, e, g, h, x, y]
e =  25  ;   (*  e = OD  by declaration   *)
g =  39  ;
h =  52  ;
f =  √(g² + h² - e²) ; (* The FOURTH BLD-segment from D to C .
```




```
(* Calculate Possible Width and Height Ranges from the
   given e, g and h values;
   Possible Ranges for: { width-range = a , height-range = b }   *)
{ { √(h² - e²) ,  h + e } , {√(g² - e²) , g + e} } // N

{{45.5961, 77.}, {29.9333, 64.}}

a  =  70;  (* Declared a-Value is possible Rectangle Width-Value. *)

(*  Break-Point Coordinate Functions : D (x,y), where  x²+y²=e²   *)
xD[e_, g_, h_] := (1/(2 a)) * (a² + e² - h²)
yD[e_, x_] := √(e² - x²)

(*  Functions for y-axis Height Point:  B (0,b)                  *)
bB[x_, y_, g_] := y + √(g² - x²)


(* Calculate: x- , y-Values for D (x,y); and b-Value for B (0,b) *)
x = xD[e, g, h] ;
y = yD[e, x]  ;
b = bB[ x, y , g ] ;

(*   Display Computed Values for: { x, y, a , b, f }             *)

{x, y, a, b, f} // N

{20.15, 14.7979, 70., 48.1892, 60.}
```

```
(*   Distance Formula Verification for: g, h and f              *)
```

$$\left\{ g - \sqrt{(b-y)^2 + x^2} \ ,\ \ h - \sqrt{(a-x)^2 + y^2} \ ,\ f - \sqrt{(b-y)^2 + (a-x)^2} \ \right\} // N$$

```
{0, 0, 0}
```

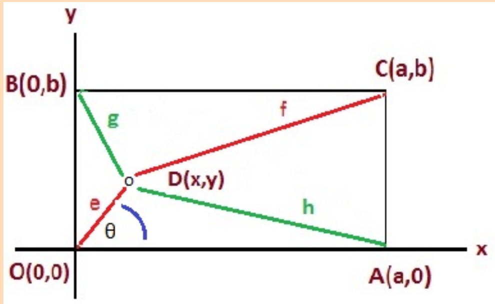
```
(*  Values   {0,0,0}  confirm the Formulated Descriptions for
    { x , y , b }  do create the correct Calculated Values
    which do compute the correct measures for the
    BLD-segments: g, h, e, and f and for the Height = b         *)
```

Computational Example: 03

(Revised Copy)

```
(*   This COMPUTATIONAL DEVELOPMENT illustrates how to determine
     if There Exists a BLD-Quadrature which will admit AND accommodate
      GIVEN DATA Measures of: e , g  and  h .
     ==========================================================
 Given:  Data Measures for:  e, g and h .
 SYMBOLS: aA,bB,zZ,xD,yD are FUNCTIONS; a,b,e,g,h,θ are REALS.  *)
Clear[aA, bB, zZ, xD, yD, a, b, e, g , h , θ]
e = 25  ;  (*  e = OD  by declaration   *)
g = 39  ;
h = 52  ;
f =  √(g² + h² - e²) ; (* The FOURTH BLD-segment from D to C .       *)
```

```
(*  Referring to Item (5.3) of Article_05, the function a (θ)
    is NOW DENOTED by aA[θ] in order to distinguish the
    PARAMETER "a" from the FUNCTION NAME "a " , etc. ; Now,...
   Examine HOW the Data affects the behavior of the strictly
   increasing function: zZ (θ) = bB (θ) - aA (θ) ;
   Note that here:  zZ (0) < g - h - e  < 0 ; and, also
   zZ (π/2) > g - h + e > 0 ; hence by Item (6.5), this Family
   of BLD Rectangles does have a BLD Quadrature.  Hence,
   now proceed to Solve the equation: zZ (θ) = 0  for  θ .
   Formulating the Mathematica Code, ...                         *)
```




```
aA[θ_] := e Cos[θ] + √(h² - ( e Sin[θ] )²) ;
bB[θ_] := e Sin[θ] + √(g² - ( e Cos[θ] )²) ;
zZ[θ_] := bB[θ] - aA[θ] ;
Solve[ zZ[θ] == 0 , θ ] // N
```

{{θ → 1.07248 }, {θ → -3.02176 }}

```
(* Use θ = 1.07278 ∈ [0, π/2] to Compute: a = aA[θ] and b= bB[θ]  *)
θ = 1.07248 ;
a = aA[θ] ;
b = bB[θ] ;
q = a     ;
{ a , b , q } // N
```

{59.0843 , 59.0842 , 59.0843 }

```
(*  Thus, a = 59.0843 = b, so:  Quadrature edge = q = 59.0843  *)
(*  Now, Formulate functions xD and yD to compute D (x,y)      *)
xD[e_, θ_] := e Cos[θ]
yD[e_, θ_] := e Sin[θ]
(*  Compute the values  { xD , yD }  for  D (x,y)              *)
x = xD[e, θ] ;
y = yD[e, θ] ;
{x, y}
```

{11.9487 , 21.9597 }

```
(* Hence ... the Coordinates of D are:
   x = 11.9487  and  y = 21.9597 .
   Now list {a, b, q, x, y}                                  *)
 {a, b, q, x, y}
```

{59.0843, 59.0842, 59.0843, 11.9487, 21.9597}

```
(* Distance Formula Verification for: g, h and f             *)
```

$$\left\{ g - \sqrt{(b-y)^2 + x^2} \;,\; h - \sqrt{(a-x)^2 + y^2} \;,\; f - \sqrt{(b-y)^2 + (a-x)^2} \right\} \text{ // N}$$

$\{0., 0., 1.42109 \times 10^{-14}\}$

```
(*  Values   {0, 0, 1.421×10^-14}  confirm the Formulated Descriptions for
   {a, b, q, x, y}  do create the correct Calculated Values
    which do compute the correct measures for the
    BLD-segments: g, h, e, and f and for the Edge Length = q   *)
```