Profiles & Perspectives of *p-Binomial Sequences*

ADDENDUM: Modeling Exercises

10/27/2018

Article 6: p-Binomial Sequence Illustrative Modeling Exercises. This article offers *Additional Exercise Models* which exemplify *actual implementations* that *illustrate* how to devise and construct *p-Binomial Sequences* $\mathbb{B}$ and/or *Linear Operators L* (the two of) which *constitute a unique, (each to the other) companion pair* $\langle L , \mathbb{B}\rangle$. The following *Modeling Exercises* devise the *Construction & Exposition* of *companion pairs* $\langle L , \mathbb{B}\rangle$ which feature *novel declarations of operators L & Q* together with *illustrations of very lengthy expansions regarding* $p_n(a{+}x)$.

Exercise 01. *Determine the number of terms* , $\#[\, p_n(a{+}x)]$, *which comprise a fully expanded p-Binomial Expansion-Sum for* $p_n(a{+}x)$.

Exploration. First, note the *binomial expansion:* $(a{+}x)^k = \sum_{j=0}^{k} \binom{k}{j} a^{k-j} x^j$; observe this sum has *(1+k)-terms*. Recall that $p_k(0) = 0,\ k > 0$; Hence, for $n > 0$: $p_n(x) = \sum_{k=1}^{n} c_k x^k \implies p_n(a{+}x) = \sum_{k=1}^{n} c_k (a{+}x)^k$. Next, note that: $\#[c_k(a{+}x)^k] = (1{+}k)$ from an above observation. Of course, *zero-value* c_k*-coefficients* contribute *null term-counts.* Consequently, for $n > 0$,

$$(1.0)\quad \#[p_n(a{+}x)] = \sum_{k=1}^{n} \#[c_k(a{+}x)^k]\,(1-\delta_0^{c_k}) = \boxed{\sum_{k=1}^{n} (1{+}k)\,(1-\delta_0^{c_k})}\ ,$$

where $\delta_i^j = \begin{cases} 0\,, \text{ if } i \neq j \\ 1, \text{ if } i = j \end{cases}$, denotes the *Kronecker delta.* □

Now suppose that p_n *exhibits all powers* x^k, $1 \le k \le n$; then, $(1-\delta_0^{c_k}) = 1$, $1 \le k \le n$; thus,

$$(1.1)\quad \#[p_n(a{+}x)] = \sum_{k=1}^{n} (1{+}k)\,(1-\delta_0^{c_k}) = \sum_{k=1}^{n} (1{+}k) = n + \tfrac{(n+1)}{2} = \boxed{\frac{n(n{+}3)}{2}}\ . \ \square$$

Exercise 02. *Suppose* that the *Operator* $Q = \sum_{k=0}^{m} \mu_k D^k$, $(\mu_k > 0,\ 0 \le k \le m)$ has *only positive* μ*-values.* Then, $Q[\,n\,p_{n-1}(t)]$ exhibits *only positive coefficients* $c_k > 0,\ 1 \le k \le n{-}1$, hence

$$(2.0)\quad p_n(x) = \int_0^x Q[\,n\,p_{n-1}(t)]\,dt \implies \#[p_n(a{+}x)] = \frac{n(n{+}3)}{2}\ ,\ 1 \le n \le m\ ,$$

for each *p-Binomial Sequence polynomial* p_n. This assertion is a *direct consequence* of the *given prescribed features of the Operator Q* and Item (1.1), above . □

Exercise 03. *Suppose that the Operator* $Q = \sum_{k=0}^{m} \mu_{2k} D^{2k}$, $(\mu_0 \neq 0)$, *is an Even-Order Operator;*

then: (i) $L = \sum_{j=0}^{n} \lambda_{2j-1} D^{2k-1}$, $(\lambda_1 \neq 0)$, *is the Odd-Order companion Operator* L;

(ii) *Odd-integer k* $\Longrightarrow$ p_k *is an odd-function polynomial;*

(iii) *Even-integer k* $\Longrightarrow$ p_k *is an even-function polynomial.*

<u>*Exploration*</u>. *Suppose the hypothesis.* Then, recall that *given* Q, *the companion* $L = \frac{D}{Q}$; now, observe that: *(i)* $L = \frac{D}{Q} = D\left(\frac{1}{Q}\right) = D \cdot [$*even-order Division Algorithm expansion of* $Q] = [$*odd-orders of* $D]$;
Observing that: $p_0(x) = 1$ and $p_k(x) = \int_0^x Q[kp_{k-1}(t)]\, dt$ *implies the following results:* (a) p_0 is an *even-function ;* (b) p_1 is an *odd-function;* (c) p_2 is an *even-function;* (d) *etc. Consequently*, the *recursive creation* of the *p-Binomial Sequence* and the *Even-Order Q-operator* render the conclusions *(i), (ii) and (iii), above.* □

Actually, ... the development of *Exercise 03* establishes that: *Given companion Operators L & Q, L is an Odd-Order Operator* $\Longleftrightarrow$ *Q is an Even-Order Operator.* Of course, due to the restrictions imposed on *each of L and Q: (a) L can not be an even-order operator; and, consequently, (b) Q can not be an odd-order operator.*

Exercise 04. *Suppose that* L *is an Odd-Order operator, so that* Q *is an Even-Order Operator.* Then, the *p-Binomial Sequence polynomials are of Odd-order, or of Even-order* as declared by the *pairty of their degree-index:* $\deg[p_k] = k$. Now, let the notation: $\#[p_k(a{+}x)]$ *denote the* <u>*number of terms*</u> *in the* <u>*complete p-Binomial Expansion-Sum*</u> *for* $p_k(a{+}x)$. Then, *applying* (1.0) *of Exercise 01:*

(i) for p_k *with* $k = 2j$ *&* <u>*no absent even-powers*</u>*:* $\#[p_k(a{+}x)] = \sum_{i=1}^{j}(1{+}2i) = j(j+2)$.

(ii) for p_n *with* $n = 2m{-}1$ *&* <u>*no absent odd-powers*</u>*:* $\#[p_n(a{+}x)] = \sum_{i=1}^{m} 2i = m(m+1)$.

Therefore, *(i) & (ii) , above*, render:

(iii) for p_k *with* ***<u>Even-k & no absent even-powers:</u>*** $\boxed{\#[p_k(a{+}x)] = \frac{k(k+4)}{4} \; , \; k > 0}$; and,

(iv) for p_n *with* ***<u>Odd-n & no absent odd-powers:</u>*** $\boxed{\#[p_n(a{+}x)] = \frac{(n+1)(n+3)}{4}}$. □

MODEL #07A. This *Model #07A* illustrates aspects of the *companion p-Binomial Sequence* $\mathbb{B}$ created by the *declared Operator* $Q = Cos(D)$; hence, this model displays the features of the *even-parity order Q* and the corresponding *Operator L* which is of *odd-parity order*. Further, illustrations using this model *verify the Formulas (iii) & (iv) of Exercise 04* regarding the number of terms which comprise the *p-Binomial Sum-Expansions* for the *elements of the Sequence* $\mathbb{B}$.

MODEL #08A. This *Model #08A* illustrates aspects of the *companion p-Binomial Sequence* $\mathbb{B}$ created by the *declared Operator* $L = Tan(D)$ *;* since L is of *odd-parity order*, then the *companion Operator Q* is of *even-parity order*. Thus, the features of the *p-Binomial Sequence* $\mathbb{B}$ for this model duplicate those of the *Model #07A, above*. Therefore, in *Model #07A*, additional illustrations using this model also *verify the Formulas (iii) & (iv) of Exercise 04* regarding the number of terms which comprise the *p-Binomial Sum-Expansions* for the *elements of the Sequence* $\mathbb{B}$ for this *Model #08A.*

Article 6. MODEL #07A.

Objective: *Declare Q = Cos[D]* integral recursion-Operator; *Create L= B_m*-companion Operator; *Create B_M= L-companion Sequence ;* *Illustrate* $Lp_k = k\, p_{k-1}$; Illustrate p-Binomial Expansion ; *#[$p_9(a+x)$] = ?*

```
Clear[d, Loper, Qoper , L , Q , p, B, L]

Qoper = Series[Cos[d], {d, 0, 18} ] (*  Qoper = Cos(x)  (declared)  *)
Loper = Series[d/Qoper , {d, 0, 18 } ]  (*  Loper devised by: LQ = D ==> L = D/Q *)
```

$$1-\frac{d^2}{2}+\frac{d^4}{24}-\frac{d^6}{720}+\frac{d^8}{40\,320}-\frac{d^{10}}{3\,628\,800}+\frac{d^{12}}{479\,001\,600}-\frac{d^{14}}{87\,178\,291\,200}+\frac{d^{16}}{20\,922\,789\,888\,000}-\frac{d^{18}}{6\,402\,373\,705\,728\,000}+O[d]^{19}$$

$$d+\frac{d^3}{2}+\frac{5\,d^5}{24}+\frac{61\,d^7}{720}+\frac{277\,d^9}{8064}+\frac{50\,521\,d^{11}}{3\,628\,800}+\frac{540\,553\,d^{13}}{95\,800\,320}+\frac{199\,360\,981\,d^{15}}{87\,178\,291\,200}+\frac{3\,878\,302\,429\,d^{17}}{4\,184\,557\,977\,600}+O[d]^{19}$$

```
μ[n_] := SeriesCoefficient[Qoper, n]  (*  pk-Recursion Q-operator coefficients  *)
λ[n_] := SeriesCoefficient[Loper, n]  (*  pk-L-operator coefficients  *)

Table[ μ[2 k], {k, 0, 9} ]
Table[ λ[2 k - 1], {k, 1, 9} ]
```

$$\left\{1,-\frac{1}{2},\frac{1}{24},-\frac{1}{720},\frac{1}{40\,320},-\frac{1}{3\,628\,800},\frac{1}{479\,001\,600},-\frac{1}{87\,178\,291\,200},\frac{1}{20\,922\,789\,888\,000},-\frac{1}{6\,402\,373\,705\,728\,000}\right\}$$

$$\left\{1,\frac{1}{2},\frac{5}{24},\frac{61}{720},\frac{277}{8064},\frac{50\,521}{3\,628\,800},\frac{540\,553}{95\,800\,320},\frac{199\,360\,981}{87\,178\,291\,200},\frac{3\,878\,302\,429}{4\,184\,557\,977\,600}\right\}$$

$$\left\{1,-\frac{1}{2},\frac{1}{24},-\frac{1}{720},\frac{1}{40\,320},-\frac{1}{3\,628\,800},\frac{1}{479\,001\,600},-\frac{1}{87\,178\,291\,200},\frac{1}{20\,922\,789\,888\,000},-\frac{1}{6\,402\,373\,705\,728\,000}\right\}$$

```
(*  μ[2k] ,  ⩽ k ≲ 9  *)
```

$$\left\{1,\frac{1}{2},\frac{5}{24},\frac{61}{720},\frac{277}{8064},\frac{50\,521}{3\,628\,800},\frac{540\,553}{95\,800\,320},\frac{199\,360\,981}{87\,178\,291\,200},\frac{3\,878\,302\,429}{4\,184\,557\,977\,600}\right\}$$

```
(*  λ[2k-1] , 1 ⩽ k ≲ 9  *)
```

$$Q[f\_] := f + \sum_{j=1}^{6} \mu[2j]\, \partial_{\{t,2j\}} f \quad (*\ \text{Operator Q } *)$$

$$L[f\_] := \sum_{k=1}^{6} \lambda[2k-1]\, \partial_{\{x,2k-1\}} f \quad (*\ \text{Operator L } *)$$

```
M = 9;  (*  M-value DELCARED for a finite p-Binomial Sequence B_M *)
p[0, x_] := 1 ;   (*  p_0(x)=1    *)
```

$$p[k\_, x\_] := \int_0^x Q[k * p[k-1, t]]\, dt \quad (*\ \text{Recursion for } p_k(x)\ *)$$

```
Print[ "p_k(x)" , "                                          " , "Lp_k(x)= kp_{k-1}(x)"]
T1 = Table[ { Expand[p[k, x] ], Factor[ L[ p[k, x] ] ]}, {k, 0, M} ];
Print[ TableForm [T1] ]  (* Sequences Table Confirms: Lp_k(x)= p_k(x)  *)
```

$p_k(x)$	$Lp_k(x) = kp_{k-1}(x)$
1	0
x	1
x^2	$2x$
$-3x + x^3$	$3x^2$
$-12x^2 + x^4$	$4x\left(-3 + x^2\right)$
$65x - 30x^3 + x^5$	$5x^2\left(-12 + x^2\right)$
$480x^2 - 60x^4 + x^6$	$6x\left(65 - 30x^2 + x^4\right)$
$-3787x + 1995x^3 - 105x^5 + x^7$	$7x^2\left(480 - 60x^2 + x^4\right)$
$-41\,216x^2 + 6160x^4 - 168x^6 + x^8$	$8x\left(-3787 + 1995x^2 - 105x^4 + x^6\right)$
$427\,905x - 242\,172x^3 + 15\,750x^5 - 252x^7 + x^9$	$9x^2\left(-41\,216 + 6160x^2 - 168x^4 + x^6\right)$

$$BiExSum[N\_] := \sum_{k=0}^{N} Binomial[N, k]\, p[N-k, a]\, p[k, x] \quad (*\ \text{p-Binomial Expansion Sum } *)$$

```
Print[ "p_k(a+x)  = { Expansion-Terms }" ]
Print["=================================================================="]
Do[ Print[ Expand[{ p[k, a + x ]}] ], {k, 0, M}]  (* p_k-sequence = { p_k(x) } *)
Print["_________________________________________________________________________"]
Print[]
Print[ "BiExSum[k]  = { Expansion-Sum Terms }" ]
Print["=================================================================="]
Do[ Print[ Expand[{ BiExSum[k]}] ], {k, 0, M}]
```

$p_k\ (a+x)\ =\ \{\text{Expansion-Terms}\}$

==

$\{1\}$

$\{a+x\}$

$\{a^2+2\,a\,x+x^2\}$

$\{-3\,a+a^3-3\,x+3\,a^2\,x+3\,a\,x^2+x^3\}$

$\{-12\,a^2+a^4-24\,a\,x+4\,a^3\,x-12\,x^2+6\,a^2\,x^2+4\,a\,x^3+x^4\}$

$\{65\,a-30\,a^3+a^5+65\,x-90\,a^2\,x+5\,a^4\,x-90\,a\,x^2+10\,a^3\,x^2-30\,x^3+10\,a^2\,x^3+5\,a\,x^4+x^5\}$

$\{480\,a^2-60\,a^4+a^6+960\,a\,x-240\,a^3\,x+6\,a^5\,x+480\,x^2-360\,a^2\,x^2+15\,a^4\,x^2-240\,a\,x^3+20\,a^3\,x^3-60\,x^4+15\,a^2\,x^4+6\,a\,x^5+x^6\}$

$\{-3787\,a+1995\,a^3-105\,a^5+a^7-3787\,x+5985\,a^2\,x-525\,a^4\,x+7\,a^6\,x+5985\,a\,x^2-1050\,a^3\,x^2+21\,a^5\,x^2+1995\,x^3-1050\,a^2\,x^3+35\,a^4\,x^3-525\,a\,x^4+35\,a^3\,x^4-105\,x^5+21\,a^2\,x^5+7\,a\,x^6+x^7\}$

$\{-41\,216\,a^2+6160\,a^4-168\,a^6+a^8-82\,432\,a\,x+24\,640\,a^3\,x-1008\,a^5\,x+8\,a^7\,x-41\,216\,x^2+36\,960\,a^2\,x^2-2520\,a^4\,x^2+28\,a^6\,x^2+24\,640\,a\,x^3-3360\,a^3\,x^3+56\,a^5\,x^3+6160\,x^4-2520\,a^2\,x^4+70\,a^4\,x^4-1008\,a\,x^5+56\,a^3\,x^5-168\,x^6+28\,a^2\,x^6+8\,a\,x^7+x^8\}$

$\{427\,905\,a-242\,172\,a^3+15\,750\,a^5-252\,a^7+a^9+427\,905\,x-726\,516\,a^2\,x+78\,750\,a^4\,x-1764\,a^6\,x+9\,a^8\,x-726\,516\,a\,x^2+157\,500\,a^3\,x^2-5292\,a^5\,x^2+36\,a^7\,x^2-242\,172\,x^3+157\,500\,a^2\,x^3-8820\,a^4\,x^3+84\,a^6\,x^3+78\,750\,a\,x^4-8820\,a^3\,x^4+126\,a^5\,x^4+15\,750\,x^5-5292\,a^2\,x^5+126\,a^4\,x^5-1764\,a\,x^6+84\,a^3\,x^6-252\,x^7+36\,a^2\,x^7+9\,a\,x^8+x^9\}$

BiExSum[k] = { Expansion-Sum Terms }

==

$\{1\}$

$\{a+x\}$

$\{a^2+2\,a\,x+x^2\}$

$\{-3\,a+a^3-3\,x+3\,a^2\,x+3\,a\,x^2+x^3\}$

$\{-12\,a^2+a^4-24\,a\,x+4\,a^3\,x-12\,x^2+6\,a^2\,x^2+4\,a\,x^3+x^4\}$

$\{65\,a-30\,a^3+a^5+65\,x-90\,a^2\,x+5\,a^4\,x-90\,a\,x^2+10\,a^3\,x^2-30\,x^3+10\,a^2\,x^3+5\,a\,x^4+x^5\}$

$\{480\,a^2-60\,a^4+a^6+960\,a\,x-240\,a^3\,x+6\,a^5\,x+480\,x^2-360\,a^2\,x^2+15\,a^4\,x^2-240\,a\,x^3+20\,a^3\,x^3-60\,x^4+15\,a^2\,x^4+6\,a\,x^5+x^6\}$

$\{-3787\,a+1995\,a^3-105\,a^5+a^7-3787\,x+5985\,a^2\,x-525\,a^4\,x+7\,a^6\,x+5985\,a\,x^2-1050\,a^3\,x^2+21\,a^5\,x^2+1995\,x^3-1050\,a^2\,x^3+35\,a^4\,x^3-525\,a\,x^4+35\,a^3\,x^4-105\,x^5+21\,a^2\,x^5+7\,a\,x^6+x^7\}$

$\{-41\,216\,a^2+6160\,a^4-168\,a^6+a^8-82\,432\,a\,x+24\,640\,a^3\,x-1008\,a^5\,x+8\,a^7\,x-41\,216\,x^2+36\,960\,a^2\,x^2-2520\,a^4\,x^2+28\,a^6\,x^2+24\,640\,a\,x^3-3360\,a^3\,x^3+56\,a^5\,x^3+6160\,x^4-2520\,a^2\,x^4+70\,a^4\,x^4-1008\,a\,x^5+56\,a^3\,x^5-168\,x^6+28\,a^2\,x^6+8\,a\,x^7+x^8\}$

$\{427\,905\,a-242\,172\,a^3+15\,750\,a^5-252\,a^7+a^9+427\,905\,x-726\,516\,a^2\,x+78\,750\,a^4\,x-1764\,a^6\,x+9\,a^8\,x-726\,516\,a\,x^2+157\,500\,a^3\,x^2-5292\,a^5\,x^2+36\,a^7\,x^2-242\,172\,x^3+157\,500\,a^2\,x^3-8820\,a^4\,x^3+84\,a^6\,x^3+78\,750\,a\,x^4-8820\,a^3\,x^4+126\,a^5\,x^4+15\,750\,x^5-5292\,a^2\,x^5+126\,a^4\,x^5-1764\,a\,x^6+84\,a^3\,x^6-252\,x^7+36\,a^2\,x^7+9\,a\,x^8+x^9\}$

(Note that this model _verifies that_: p_k(a+x) = BiExSum[k] for each $0 \leqslant k \leqslant 9$)

Also note that: (1) #[p[9 , a+x]] = $\frac{(9+1)(9+3)}{4}$ = 30 (# of terms comprising the expansion of: $p_9(a+x)$) ; and,

(2) #[p[8 , a+x]] = $\frac{(8)(8+4)}{4}$ = 24 (# of terms comprising the expansion of: $p_8(a+x)$) ;

by appealing the _number of terms formulae_ *displayed in the* _ADDENDUM developments_ ; Exercise 04.

Article 6. MODEL #08A.

Objective: *Declare L = Tan[D]; Devise Q; Create p_k –sequence B_M ; Illustrate* $Lp_k = k\, p_{k-1}$; Model number of terms: p_{28}(a+x) expansion.

```
Clear[d, Loper, Qoper , L , Q , p, B, L]
```

```
Loper = Series[Tan[d], {d, 0 , 30} ] (*  L declared as L = Tan(D) *)
Qoper = Series[d / Loper , {d, 0, 30} ] (*  Q devised as:  Q = D/L   *)
```

$$d+\frac{d^3}{3}+\frac{2 d^5}{15}+\frac{17 d^7}{315}+\frac{62 d^9}{2835}+\frac{1382 d^{11}}{155925}+\frac{21844 d^{13}}{6081075}+\frac{929569 d^{15}}{638512875}+\frac{6404582 d^{17}}{10854718875}+\frac{443861162 d^{19}}{1856156927625}+\frac{18888466084 d^{21}}{194896477400625}+\frac{113927491862 d^{23}}{2900518163668125}+\frac{58870668456604 d^{25}}{3698160658676859375}+\frac{8374643517010684 d^{27}}{1298054391195577640625}+\frac{689005380505609448 d^{29}}{263505041412702261046875}+O[d]^{31}$$

$$1-\frac{d^2}{3}-\frac{d^4}{45}-\frac{2 d^6}{945}-\frac{d^8}{4725}-\frac{2 d^{10}}{93555}-\frac{1382 d^{12}}{638512875}-\frac{4 d^{14}}{18243225}-\frac{3617 d^{16}}{162820783125}-\frac{87734 d^{18}}{38979295480125}-\frac{349222 d^{20}}{1531329465290625}-\frac{310732 d^{22}}{13447856940643125}-\frac{472728182 d^{24}}{201919571963756521875}-\frac{2631724 d^{26}}{11094481976030578125}-\frac{13571120588 d^{28}}{564653660170076273671875}+O[d]^{30}$$

```
λ[n_] := SeriesCoefficient[Loper, n] (*  Coefficients for L  *)
μ[n_] := SeriesCoefficient[Qoper, n]  (*  Coefficients for Q  *)
```

```
Table[ λ[2 k - 1] , {k, 1, 14} ]
Table[ μ[2 k] , {k, 0, 14} ]
```

$$\left\{1,\frac{1}{3},\frac{2}{15},\frac{17}{315},\frac{62}{2835},\frac{1382}{155925},\frac{21844}{6081075},\frac{929569}{638512875},\frac{6404582}{10854718875},\frac{443861162}{1856156927625},\frac{18888466084}{194896477400625},\frac{113927491862}{2900518163668125},\frac{58870668456604}{3698160658676859375},\frac{8374643517010684}{1298054391195577640625}\right\}$$

$$\left\{1,-\frac{1}{3},-\frac{1}{45},-\frac{2}{945},-\frac{1}{4725},-\frac{2}{93555},-\frac{1382}{638512875},-\frac{4}{18243225},-\frac{3617}{162820783125},-\frac{87734}{38979295480125},-\frac{349222}{1531329465290625},-\frac{310732}{13447856940643125},-\frac{472728182}{201919571963756521875},-\frac{2631724}{11094481976030578125},-\frac{13571120588}{564653660170076273671875}\right\}$$

```
L[f_] := Sum[λ[2 k - 1] ∂_{x,2 k-1} f, {k, 1, n}]

Q[f_] := f + Sum[μ[2 k] ∂_{t,2 k} f, {k, 1, n}]

n = 14;  m = 28 ;
p[0, x_] := 1
p[k_, x_] := Integrate[Q [ k p[k - 1, t] ], {t, 0, x}]
```

```
Print[  "TABLE DISPLAYING:   pk(x)  "]
Do[  Print [ Factor [ p[k, x] ] ],   {k, 0, m} ];  (*  p-Binomial Sequence = Bm  *)
Print[]
Print["TABLE DISPLAYING:   Lpk(x)  ;   Note That:   Lpk(x)  =  kpk-1(x)  "]
Do[  Print [ Factor [ L[ p[k, x]] ] ],    {k, 0, m} ];
```

TABLE DISPLAYING: $p_k(x)$

1

x

x^2

$x\left(-2+x^2\right)$

$x^2\left(-8+x^2\right)$

$x\left(24-20x^2+x^4\right)$

$x^2\left(184-40x^2+x^4\right)$

$x\left(-720+784x^2-70x^4+x^6\right)$

$x^2\left(-24+x^2\right)\left(352-88x^2+x^4\right)$

$x\left(40320-52352x^2+6384x^4-168x^6+x^8\right)$

$x^2\left(648576-229760x^2+14448x^4-240x^6+x^8\right)$

$x\left(-3628800+5360256x^2-804320x^4+29568x^6-330x^8+x^{10}\right)$

$x^2\left(-74972160+30633856x^2-2393600x^4+55968x^6-440x^8+x^{10}\right)$

$x\left(479001600-782525952x^2+136804096x^4-6296576x^6+99528x^8-572x^{10}+x^{12}\right)$

$x^2\left(12174658560-5561407488x^2+510205696x^4-15027584x^6+168168x^8-728x^{10}+x^{12}\right)$

$x\left(-87178291200+154594381824x^2-30459752960x^4+1656182528x^6-33141680x^8+272272x^{10}-910x^{12}+x^{14}\right)$

$x^2\left(-2643856588800+1322489954304x^2-137602949120x^4+4811975168x^6-68456960x^8+425152x^{10}-1120x^{12}+x^{14}\right)$

$x\left(20922789888000-39746508226560x^2+8632830664704x^4-535086755840x^6+12765978368x^8-133802240x^{10}+643552x^{12}-1360x^{14}+x^{16}\right)$

$x^2\left(740051782041600-399463775797248x^2+46060832825344x^4-1843944001536x^6+31386271488x^8-249443584x^{10}+948192x^{12}-1632x^{14}+x^{16}\right)$

$x\left(-6402373705728000+12902483299368960x^2-3041109959196672x^4+209797380112384x^6-5750333382144x^8+72329756928x^{10}-446370496x^{12}+1364352x^{14}-1938x^{16}+x^{18}\right)$

$$x^2 \left(-259 500 083 163 955 200 + 149 519 094 622 027 776 x^2 - 18 793 914 785 464 320 x^4 + 840 426 228 637 696 x^6 - 16 484 438 231 040 x^8 + 157 639 462 656 x^{10} - 770 652 160 x^{12} + 1 922 496 x^{14} - 2280 x^{16} + x^{18}\right)$$

$$x \left(2 432 902 008 176 640 000 - 5 162 443 736 924 160 000 x^2 + 1 305 140 879 116 763 136 x^4 - 98 516 919 228 170 240 x^6 + 3 025 552 913 852 416 x^8 - 43 969 745 863 680 x^{10} + 327 260 251 136 x^{12} - 1 289 105 920 x^{14} + 2 658 936 x^{16} - 2660 x^{18} + x^{20}\right)$$

$$x^2 \left(111 422 936 937 037 824 000 - 67 960 463 478 175 825 920 x^2 + 9 198 585 089 011 777 536 x^4 - 451 495 935 256 002 560 x^6 + 9 949 016 970 889 216 x^8 - 110 178 320 179 200 x^{10} + 650 934 158 336 x^{12} - 2 096 554 240 x^{14} + 3 616 536 x^{16} - 3080 x^{18} + x^{20}\right)$$

$$x \left(-1 124 000 727 777 607 680 000 + 2 496 471 943 395 999 744 000 x^2 - 670 935 549 630 120 394 752 x^4 + 54 713 401 772 426 428 416 x^6 - 1 849 301 381 455 818 752 x^8 + 30 263 039 559 909 376 x^{10} - 261 372 556 204 032 x^{12} + 1 246 501 093 376 x^{14} - 3 324 982 672 x^{16} + 4 845 456 x^{18} - 3542 x^{20} + x^{22}\right)$$

$$x^2 \left(-57 504 006 817 918 746 624 000 + 36 884 466 463 352 967 659 520 x^2 - 5 325 419 604 670 079 827 968 x^4 + 283 170 345 011 963 723 776 x^6 - 6 883 503 968 725 762 048 x^8 + 86 013 269 570 584 576 x^{10} - 590 745 240 322 048 x^{12} + 2 307 357 538 816 x^{14} - 5 154 949 888 x^{16} + 6 403 936 x^{18} - 4048 x^{20} + x^{22}\right)$$

$$x \left(620 448 401 733 239 439 360 000 - 1 435 556 519 572 510 605 312 000 x^2 + 407 240 889 859 179 425 562 624 x^4 - 35 527 217 383 049 468 313 600 x^6 + 1 303 984 707 575 575 674 880 x^8 - 23 588 701 805 795 737 600 x^{10} + 230 290 920 595 210 240 x^{12} - 1 278 813 843 865 600 x^{14} + 4 142 747 973 760 x^{16} - 7 829 641 600 x^{18} + 8 359 120 x^{20} - 4600 x^{22} + x^{24}\right)$$

$$x^2 \left(35 122 852 492 484 487 413 760 000 - 23 566 236 397 584 291 201 024 000 x^2 + 3 602 492 652 661 227 322 343 424 x^4 - 205 429 424 390 227 702 579 200 x^6 + 5 434 087 088 811 032 903 680 x^8 - 75 196 663 548 146 483 200 x^{10} + 584 738 064 788 439 040 x^{12} - 2 663 228 367 155 200 x^{14} + 7 235 717 906 560 x^{16} - 11 672 003 200 x^{18} + 10 787 920 x^{20} - 5200 x^{22} + x^{24}\right)$$

$$x \left(-403 291 461 126 605 635 584 000 000 + 968 234 590 214 616 380 866 560 000 x^2 - 288 272 814 806 050 917 816 729 600 x^4 + 26 695 183 951 643 381 726 183 424 x^6 - 1 053 019 484 314 351 891 251 200 x^8 + 20 766 743 262 578 262 343 680 x^{10} - 224 885 761 935 033 139 200 x^{12} + 1 415 967 063 301 079 040 x^{14} - 5 356 014 550 099 200 x^{16} + 12 324 984 946 560 x^{18} - 17 105 431 200 x^{20} + 13 777 920 x^{22} - 5850 x^{24} + x^{26}\right)$$

$$x^2 \left(-25 059 533 910 850 715 800 043 520 000 + 17 511 732 541 318 788 803 985 408 000 x^2 - 2 817 222 656 974 232 498 101 813 248 x^4 + 170 906 639 873 583 228 936 781 824 x^6 - 4 867 748 620 659 696 989 634 560 x^8 + 73 554 801 073 377 093 550 080 x^{10} - 635 371 883 416 517 345 280 x^{12} + 3 285 553 377 044 029 440 x^{14} - 10 435 488 520 504 320 x^{16} + 20 518 731 192 960 x^{18} - 24 678 551 040 x^{20} + 17 428 320 x^{22} - 6552 x^{24} + x^{26}\right)$$

<u>TABLE</u> <u>DISPLAYING</u>: $Lp_k(x)$; <u>Note</u> <u>That</u>: $Lp_k(x) = k p_{k-1}(x)$

0

1

$2 x$

$3 x^2$

$4 x \left(-2 + x^2\right)$

$5 x^2 \left(-8 + x^2\right)$

$6 x \left(24 - 20 x^2 + x^4\right)$

$7\ x^2\ (184 - 40\ x^2 + x^4)$

$8\ x\ (-720 + 784\ x^2 - 70\ x^4 + x^6)$

$9\ x^2\ (-24 + x^2)\ (352 - 88\ x^2 + x^4)$

$10\ x\ (40\,320 - 52\,352\ x^2 + 6384\ x^4 - 168\ x^6 + x^8)$

$11\ x^2\ (648\,576 - 229\,760\ x^2 + 14\,448\ x^4 - 240\ x^6 + x^8)$

$12\ x\ (-3\,628\,800 + 5\,360\,256\ x^2 - 804\,320\ x^4 + 29\,568\ x^6 - 330\ x^8 + x^{10})$

$13\ x^2\ (-74\,972\,160 + 30\,633\,856\ x^2 - 2\,393\,600\ x^4 + 55\,968\ x^6 - 440\ x^8 + x^{10})$

$14\ x\ (479\,001\,600 - 782\,525\,952\ x^2 + 136\,804\,096\ x^4 - 6\,296\,576\ x^6 + 99\,528\ x^8 - 572\ x^{10} + x^{12})$

$15\ x^2\ (12\,174\,658\,560 - 5\,561\,407\,488\ x^2 + 510\,205\,696\ x^4 - 15\,027\,584\ x^6 + 168\,168\ x^8 - 728\ x^{10} + x^{12})$

$16\ x\ (-87\,178\,291\,200 + 154\,594\,381\,824\ x^2 - 30\,459\,752\,960\ x^4 + 1\,656\,182\,528\ x^6 - 33\,141\,680\ x^8 + 272\,272\ x^{10} - 910\ x^{12} + x^{14})$

$17\ x^2\ (-2\,643\,856\,588\,800 + 1\,322\,489\,954\,304\ x^2 - 137\,602\,949\,120\ x^4 + 4\,811\,975\,168\ x^6 - 68\,456\,960\ x^8 + 425\,152\ x^{10} - 1120\ x^{12} + x^{14})$

$18\ x\ (20\,922\,789\,888\,000 - 39\,746\,508\,226\,560\ x^2 + 8\,632\,830\,664\,704\ x^4 - 535\,086\,755\,840\ x^6 + 12\,765\,978\,368\ x^8 - 133\,802\,240\ x^{10} + 643\,552\ x^{12} - 1360\ x^{14} + x^{16})$

$19\ x^2\ (740\,051\,782\,041\,600 - 399\,463\,775\,797\,248\ x^2 + 46\,060\,832\,825\,344\ x^4 - 1\,843\,944\,001\,536\ x^6 + 31\,386\,271\,488\ x^8 - 249\,443\,584\ x^{10} + 948\,192\ x^{12} - 1632\ x^{14} + x^{16})$

$20\ x\ (-6\,402\,373\,705\,728\,000 + 12\,902\,483\,299\,368\,960\ x^2 - 3\,041\,109\,959\,196\,672\ x^4 + 209\,797\,380\,112\,384\ x^6 - 5\,750\,333\,382\,144\ x^8 + 72\,329\,756\,928\ x^{10} - 446\,370\,496\ x^{12} + 1\,364\,352\ x^{14} - 1938\ x^{16} + x^{18})$

$21\ x^2\ (-259\,500\,083\,163\,955\,200 + 149\,519\,094\,622\,027\,776\ x^2 - 18\,793\,914\,785\,464\,320\ x^4 + 840\,426\,228\,637\,696\ x^6 - 16\,484\,438\,231\,040\ x^8 + 157\,639\,462\,656\ x^{10} - 770\,652\,160\ x^{12} + 1\,922\,496\ x^{14} - 2280\ x^{16} + x^{18})$

$22\ x\ (2\,432\,902\,008\,176\,640\,000 - 5\,162\,443\,736\,924\,160\,000\ x^2 + 1\,305\,140\,879\,116\,763\,136\ x^4 - 98\,516\,919\,228\,170\,240\ x^6 + 3\,025\,552\,913\,852\,416\ x^8 - 43\,969\,745\,863\,680\ x^{10} + 327\,260\,251\,136\ x^{12} - 1\,289\,105\,920\ x^{14} + 2\,658\,936\ x^{16} - 2660\ x^{18} + x^{20})$

$23\ x^2\ (111\,422\,936\,937\,037\,824\,000 - 67\,960\,463\,478\,175\,825\,920\ x^2 + 9\,198\,585\,089\,011\,777\,536\ x^4 - 451\,495\,935\,256\,002\,560\ x^6 + 9\,949\,016\,970\,889\,216\ x^8 - 110\,178\,320\,179\,200\ x^{10} + 650\,934\,158\,336\ x^{12} - 2\,096\,554\,240\ x^{14} + 3\,616\,536\ x^{16} - 3080\ x^{18} + x^{20})$

$24\ x\ (-1\,124\,000\,727\,777\,607\,680\,000 + 2\,496\,471\,943\,395\,999\,744\,000\ x^2 - 670\,935\,549\,630\,120\,394\,752\ x^4 + 54\,713\,401\,772\,426\,428\,416\ x^6 - 1\,849\,301\,381\,455\,818\,752\ x^8 + 30\,263\,039\,559\,909\,376\ x^{10} - 261\,372\,556\,204\,032\ x^{12} + 1\,246\,501\,093\,376\ x^{14} - 3\,324\,982\,672\ x^{16} + 4\,845\,456\ x^{18} - 3542\ x^{20} + x^{22})$

$25\ x^2\ (-57\,504\,006\,817\,918\,746\,624\,000 + 36\,884\,466\,463\,352\,967\,659\,520\ x^2 - 5\,325\,419\,604\,670\,079\,827\,968\ x^4 + 283\,170\,345\,011\,963\,723\,776\ x^6 - 6\,883\,503\,968\,725\,762\,048\ x^8 + 86\,013\,269\,570\,584\,576\ x^{10} - 590\,745\,240\,322\,048\ x^{12} + 2\,307\,357\,538\,816\ x^{14} - 5\,154\,949\,888\ x^{16} + 6\,403\,936\ x^{18} - 4048\ x^{20} + x^{22})$

$26\ x\ (620\,448\,401\,733\,239\,439\,360\,000 - 1\,435\,556\,519\,572\,510\,605\,312\,000\ x^2 + 407\,240\,889\,859\,179\,425\,562\,624\ x^4 - 35\,527\,217\,383\,049\,468\,313\,600\ x^6 + 1\,303\,984\,707\,575\,575\,674\,880\ x^8 - 23\,588\,701\,805\,795\,737\,600\ x^{10} + 230\,290\,920\,595\,210\,240\ x^{12} - 1\,278\,813\,843\,865\,600\ x^{14} + 4\,142\,747\,973\,760\ x^{16} - 7\,829\,641\,600\ x^{18} + 8\,359\,120\ x^{20} - 4600\ x^{22} + x^{24})$

$27\ x^2\ (35\,122\,852\,492\,484\,487\,413\,760\,000 - 23\,566\,236\,397\,584\,291\,201\,024\,000\ x^2 + 3\,602\,492\,652\,661\,227\,322\,343\,424\ x^4 - 205\,429\,424\,390\,227\,702\,579\,200\ x^6 + 5\,434\,087\,088\,811\,032\,903\,680\ x^8 - 75\,196\,663\,548\,146\,483\,200\ x^{10} + 584\,738\,064\,788\,439\,040\ x^{12} - 2\,663\,228\,367\,155\,200\ x^{14} + 7\,235\,717\,906\,560\ x^{16} - 11\,672\,003\,200\ x^{18} + 10\,787\,920\ x^{20} - 5200\ x^{22} + x^{24})$

$28\,x\,\big(-403\,291\,461\,126\,605\,635\,584\,000\,000 + 968\,234\,590\,214\,616\,380\,866\,560\,000\,x^{2} -$
$288\,272\,814\,806\,050\,917\,816\,729\,600\,x^{4} + 26\,695\,183\,951\,643\,381\,726\,183\,424\,x^{6} -$
$1\,053\,019\,484\,314\,351\,891\,251\,200\,x^{8} + 20\,766\,743\,262\,578\,262\,343\,680\,x^{10} -$
$224\,885\,761\,935\,033\,139\,200\,x^{12} + 1\,415\,967\,063\,301\,079\,040\,x^{14} - 5\,356\,014\,550\,099\,200\,x^{16} +$
$12\,324\,984\,946\,560\,x^{18} - 17\,105\,431\,200\,x^{20} + 13\,777\,920\,x^{22} - 5850\,x^{24} + x^{26}\big)$

```
Expand[ p[28, a+x] ]
(*  this expansion renders 224 Terms = n(n+4)/4 using # of terms formula ; here:  n=28 (even parity) *)
```

In[ ]:=

$- 25\,059\,533\,910\,850\,715\,800\,043\,520\,000\,a^{2} + 17\,511\,732\,541\,318\,788\,803\,985\,408\,000\,a^{4} -$
$2\,817\,222\,656\,974\,232\,498\,101\,813\,248\,a^{6} + 170\,906\,639\,873\,583\,228\,936\,781\,824\,a^{8} -$
$4\,867\,748\,620\,659\,696\,989\,634\,560\,a^{10} + 73\,554\,801\,073\,377\,093\,550\,080\,a^{12} -$
$635\,371\,883\,416\,517\,345\,280\,a^{14} + 3\,285\,553\,377\,044\,029\,440\,a^{16} - 10\,435\,488\,520\,504\,320\,a^{18} +$
$20\,518\,731\,192\,960\,a^{20} - 24\,678\,551\,040\,a^{22} + 17\,428\,320\,a^{24} - 6552\,a^{26} + a^{28} -$
$50\,119\,067\,821\,701\,431\,600\,087\,040\,000\,a\,x + 70\,046\,930\,165\,275\,155\,215\,941\,632\,000\,a^{3}\,x -$
$16\,903\,335\,941\,845\,394\,988\,610\,879\,488\,a^{5}\,x + 1\,367\,253\,118\,988\,665\,831\,494\,254\,592\,a^{7}\,x -$
$48\,677\,486\,206\,596\,969\,896\,345\,600\,a^{9}\,x + 882\,657\,612\,880\,525\,122\,600\,960\,a^{11}\,x -$
$8\,895\,206\,367\,831\,242\,833\,920\,a^{13}\,x + 52\,568\,854\,032\,704\,471\,040\,a^{15}\,x - 187\,838\,793\,369\,077\,760\,a^{17}\,x +$
$410\,374\,623\,859\,200\,a^{19}\,x - 542\,928\,122\,880\,a^{21}\,x + 418\,279\,680\,a^{23}\,x - 170\,352\,a^{25}\,x + 28\,a^{27}\,x -$
$25\,059\,533\,910\,850\,715\,800\,043\,520\,000\,x^{2} + 105\,070\,395\,247\,912\,732\,823\,912\,448\,000\,a^{2}\,x^{2} -$
$42\,258\,339\,854\,613\,487\,471\,527\,198\,720\,a^{4}\,x^{2} + 4\,785\,385\,916\,460\,330\,410\,229\,891\,072\,a^{6}\,x^{2} -$
$219\,048\,687\,929\,686\,364\,533\,555\,200\,a^{8}\,x^{2} + 4\,854\,616\,870\,842\,888\,174\,305\,280\,a^{10}\,x^{2} -$
$57\,818\,841\,390\,903\,078\,420\,480\,a^{12}\,x^{2} + 394\,266\,405\,245\,283\,532\,800\,a^{14}\,x^{2} -$
$1\,596\,629\,743\,637\,160\,960\,a^{16}\,x^{2} + 3\,898\,558\,926\,662\,400\,a^{18}\,x^{2} - 5\,700\,745\,290\,240\,a^{20}\,x^{2} +$
$4\,810\,216\,320\,a^{22}\,x^{2} - 2\,129\,400\,a^{24}\,x^{2} + 378\,a^{26}\,x^{2} + 70\,046\,930\,165\,275\,155\,215\,941\,632\,000\,a\,x^{3} -$
$56\,344\,453\,139\,484\,649\,962\,036\,264\,960\,a^{3}\,x^{3} + 9\,570\,771\,832\,920\,660\,820\,459\,782\,144\,a^{5}\,x^{3} -$
$584\,129\,834\,479\,163\,638\,756\,147\,200\,a^{7}\,x^{3} + 16\,182\,056\,236\,142\,960\,581\,017\,600\,a^{9}\,x^{3} -$
$231\,275\,365\,563\,612\,313\,681\,920\,a^{11}\,x^{3} + 1\,839\,909\,891\,144\,656\,486\,400\,a^{13}\,x^{3} -$
$8\,515\,358\,632\,731\,525\,120\,a^{15}\,x^{3} + 23\,391\,353\,559\,974\,400\,a^{17}\,x^{3} - 38\,004\,968\,601\,600\,a^{19}\,x^{3} +$
$35\,274\,919\,680\,a^{21}\,x^{3} - 17\,035\,200\,a^{23}\,x^{3} + 3276\,a^{25}\,x^{3} + 17\,511\,732\,541\,318\,788\,803\,985\,408\,000\,x^{4} -$
$42\,258\,339\,854\,613\,487\,471\,527\,198\,720\,a^{2}\,x^{4} + 11\,963\,464\,791\,150\,826\,025\,574\,727\,680\,a^{4}\,x^{4} -$
$1\,022\,227\,210\,338\,536\,367\,823\,257\,600\,a^{6}\,x^{4} + 36\,409\,626\,531\,321\,661\,307\,289\,600\,a^{8}\,x^{4} -$
$636\,007\,255\,299\,933\,862\,625\,280\,a^{10}\,x^{4} + 5\,979\,707\,146\,220\,133\,580\,800\,a^{12}\,x^{4} -$
$31\,932\,594\,872\,743\,219\,200\,a^{14}\,x^{4} + 99\,413\,252\,629\,891\,200\,a^{16}\,x^{4} - 180\,523\,600\,857\,600\,a^{18}\,x^{4} +$
$185\,193\,328\,320\,a^{20}\,x^{4} - 97\,952\,400\,a^{22}\,x^{4} + 20\,475\,a^{24}\,x^{4} - 16\,903\,335\,941\,845\,394\,988\,610\,879\,488\,a\,x^{5} +$
$9\,570\,771\,832\,920\,660\,820\,459\,782\,144\,a^{3}\,x^{5} - 1\,226\,672\,652\,406\,243\,641\,387\,909\,120\,a^{5}\,x^{5} +$
$58\,255\,402\,450\,114\,658\,091\,663\,360\,a^{7}\,x^{5} - 1\,272\,014\,510\,599\,867\,725\,250\,560\,a^{9}\,x^{5} +$
$14\,351\,297\,150\,928\,320\,593\,920\,a^{11}\,x^{5} - 89\,411\,265\,643\,681\,013\,760\,a^{13}\,x^{5} +$
$318\,122\,408\,415\,651\,840\,a^{15}\,x^{5} - 649\,884\,963\,087\,360\,a^{17}\,x^{5} + 740\,773\,313\,280\,a^{19}\,x^{5} -$
$430\,990\,560\,a^{21}\,x^{5} + 98\,280\,a^{23}\,x^{5} - 2\,817\,222\,656\,974\,232\,498\,101\,813\,248\,x^{6} +$
$4\,785\,385\,916\,460\,330\,410\,229\,891\,072\,a^{2}\,x^{6} - 1\,022\,227\,210\,338\,536\,367\,823\,257\,600\,a^{4}\,x^{6} +$
$67\,964\,636\,191\,800\,434\,440\,273\,920\,a^{6}\,x^{6} - 1\,908\,021\,765\,899\,801\,587\,875\,840\,a^{8}\,x^{6} +$
$26\,310\,711\,443\,368\,587\,755\,520\,a^{10}\,x^{6} - 193\,724\,408\,894\,642\,196\,480\,a^{12}\,x^{6} +$
$795\,306\,021\,039\,129\,600\,a^{14}\,x^{6} - 1\,841\,340\,728\,747\,520\,a^{16}\,x^{6} + 2\,345\,782\,158\,720\,a^{18}\,x^{6} -$
$1\,508\,466\,960\,a^{20}\,x^{6} + 376\,740\,a^{22}\,x^{6} + 1\,367\,253\,118\,988\,665\,831\,494\,254\,592\,a\,x^{7} -$
$584\,129\,834\,479\,163\,638\,756\,147\,200\,a^{3}\,x^{7} + 58\,255\,402\,450\,114\,658\,091\,663\,360\,a^{5}\,x^{7} -$
$2\,180\,596\,303\,885\,487\,529\,000\,960\,a^{7}\,x^{7} + 37\,586\,730\,633\,383\,696\,793\,600\,a^{9}\,x^{7} -$
$332\,098\,986\,676\,529\,479\,680\,a^{11}\,x^{7} + 1\,590\,612\,042\,078\,259\,200\,a^{13}\,x^{7} - 4\,208\,778\,808\,565\,760\,a^{15}\,x^{7}$

$+ 6\,032\,011\,265\,280\,a^{17}\,x^{7} - 4\,309\,905\,600\,a^{19}\,x^{7} + 1\,184\,040\,a^{21}\,x^{7} +$

$170\,906\,639\,873\,583\,228\,936\,781\,824\,x^{8} - 219\,048\,687\,929\,686\,364\,533\,555\,200\,a^{2}\,x^{8} +$
$36\,409\,626\,531\,321\,661\,307\,289\,600\,a^{4}\,x^{8} - 1\,908\,021\,765\,899\,801\,587\,875\,840\,a^{6}\,x^{8} +$
$42\,285\,071\,962\,556\,658\,892\,800\,a^{8}\,x^{8} - 456\,636\,106\,680\,228\,034\,560\,a^{10}\,x^{8} +$
$2\,584\,744\,568\,377\,171\,200\,a^{12}\,x^{8} - 7\,891\,460\,266\,060\,800\,a^{14}\,x^{8} + 12\,818\,023\,938\,720\,a^{16}\,x^{8} -$
$10\,236\,025\,800\,a^{18}\,x^{8} + 3\,108\,105\,a^{20}\,x^{8} - 48\,677\,486\,206\,596\,969\,896\,345\,600\,a\,x^{9} +$
$16\,182\,056\,236\,142\,960\,581\,017\,600\,a^{3}\,x^{9} - 1\,272\,014\,510\,599\,867\,725\,250\,560\,a^{5}\,x^{9} +$
$37\,586\,730\,633\,383\,696\,793\,600\,a^{7}\,x^{9} - 507\,373\,451\,866\,920\,038\,400\,a^{9}\,x^{9} +$
$3\,446\,326\,091\,169\,561\,600\,a^{11}\,x^{9} - 12\,275\,604\,858\,316\,800\,a^{13}\,x^{9} + 22\,787\,598\,113\,280\,a^{15}\,x^{9} -$
$20\,472\,051\,600\,a^{17}\,x^{9} + 6\,906\,900\,a^{19}\,x^{9} - 4\,867\,748\,620\,659\,696\,989\,634\,560\,x^{10} +$
$4\,854\,616\,870\,842\,888\,174\,305\,280\,a^{2}\,x^{10} - 636\,007\,255\,299\,933\,862\,625\,280\,a^{4}\,x^{10} +$
$26\,310\,711\,443\,368\,587\,755\,520\,a^{6}\,x^{10} - 456\,636\,106\,680\,228\,034\,560\,a^{8}\,x^{10} +$
$3\,790\,958\,700\,286\,517\,760\,a^{10}\,x^{10} - 15\,958\,286\,315\,811\,840\,a^{12}\,x^{10} + 34\,181\,397\,169\,920\,a^{14}\,x^{10} -$
$34\,802\,487\,720\,a^{16}\,x^{10} + 13\,123\,110\,a^{18}\,x^{10} + 882\,657\,612\,880\,525\,122\,600\,960\,a\,x^{11} -$
$231\,275\,365\,563\,612\,313\,681\,920\,a^{3}\,x^{11} + 14\,351\,297\,150\,928\,320\,593\,920\,a^{5}\,x^{11} -$
$332\,098\,986\,676\,529\,479\,680\,a^{7}\,x^{11} + 3\,446\,326\,091\,169\,561\,600\,a^{9}\,x^{11} -$
$17\,409\,039\,617\,249\,280\,a^{11}\,x^{11} + 43\,503\,596\,398\,080\,a^{13}\,x^{11} - 50\,621\,800\,320\,a^{15}\,x^{11} +$
$21\,474\,180\,a^{17}\,x^{11} + 73\,554\,801\,073\,377\,093\,550\,080\,x^{12} - 57\,818\,841\,390\,903\,078\,420\,480\,a^{2}\,x^{12} +$
$5\,979\,707\,146\,220\,133\,580\,800\,a^{4}\,x^{12} - 193\,724\,408\,894\,642\,196\,480\,a^{6}\,x^{12} +$
$2\,584\,744\,568\,377\,171\,200\,a^{8}\,x^{12} - 15\,958\,286\,315\,811\,840\,a^{10}\,x^{12} + 47\,128\,896\,097\,920\,a^{12}\,x^{12} -$
$63\,277\,250\,400\,a^{14}\,x^{12} + 30\,421\,755\,a^{16}\,x^{12} - 8\,895\,206\,367\,831\,242\,833\,920\,a\,x^{13} +$
$1\,839\,909\,891\,144\,656\,486\,400\,a^{3}\,x^{13} - 89\,411\,265\,643\,681\,013\,760\,a^{5}\,x^{13} +$
$1\,590\,612\,042\,078\,259\,200\,a^{7}\,x^{13} - 12\,275\,604\,858\,316\,800\,a^{9}\,x^{13} + 43\,503\,596\,398\,080\,a^{11}\,x^{13} -$
$68\,144\,731\,200\,a^{13}\,x^{13} + 37\,442\,160\,a^{15}\,x^{13} - 635\,371\,883\,416\,517\,345\,280\,x^{14} +$
$394\,266\,405\,245\,283\,532\,800\,a^{2}\,x^{14} - 31\,932\,594\,872\,743\,219\,200\,a^{4}\,x^{14} + 795\,306\,021\,039\,129\,600\,a^{6}\,x^{14} -$
$7\,891\,460\,266\,060\,800\,a^{8}\,x^{14} + 34\,181\,397\,169\,920\,a^{10}\,x^{14} - 63\,277\,250\,400\,a^{12}\,x^{14} + 40\,116\,600\,a^{14}\,x^{14} +$
$52\,568\,854\,032\,704\,471\,040\,a\,x^{15} - 8\,515\,358\,632\,731\,525\,120\,a^{3}\,x^{15} + 318\,122\,408\,415\,651\,840\,a^{5}\,x^{15} -$
$4\,208\,778\,808\,565\,760\,a^{7}\,x^{15} + 22\,787\,598\,113\,280\,a^{9}\,x^{15} - 50\,621\,800\,320\,a^{11}\,x^{15} + 37\,442\,160\,a^{13}\,x^{15} +$
$3\,285\,553\,377\,044\,029\,440\,x^{16} - 1\,596\,629\,743\,637\,160\,960\,a^{2}\,x^{16} + 99\,413\,252\,629\,891\,200\,a^{4}\,x^{16} -$
$1\,841\,340\,728\,747\,520\,a^{6}\,x^{16} + 12\,818\,023\,938\,720\,a^{8}\,x^{16} - 34\,802\,487\,720\,a^{10}\,x^{16} +$
$30\,421\,755\,a^{12}\,x^{16} - 187\,838\,793\,369\,077\,760\,a\,x^{17} + 23\,391\,353\,559\,974\,400\,a^{3}\,x^{17} -$
$649\,884\,963\,087\,360\,a^{5}\,x^{17} + 6\,032\,011\,265\,280\,a^{7}\,x^{17} - 20\,472\,051\,600\,a^{9}\,x^{17} + 21\,474\,180\,a^{11}\,x^{17} -$
$10\,435\,488\,520\,504\,320\,x^{18} + 3\,898\,558\,926\,662\,400\,a^{2}\,x^{18} - 180\,523\,600\,857\,600\,a^{4}\,x^{18} +$
$2\,345\,782\,158\,720\,a^{6}\,x^{18} - 10\,236\,025\,800\,a^{8}\,x^{18} + 13\,123\,110\,a^{10}\,x^{18} + 410\,374\,623\,859\,200\,a\,x^{19} -$
$38\,004\,968\,601\,600\,a^{3}\,x^{19} + 740\,773\,313\,280\,a^{5}\,x^{19} - 4\,309\,905\,600\,a^{7}\,x^{19} + 6\,906\,900\,a^{9}\,x^{19} +$
$20\,518\,731\,192\,960\,x^{20} - 5\,700\,745\,290\,240\,a^{2}\,x^{20} + 185\,193\,328\,320\,a^{4}\,x^{20} - 1\,508\,466\,960\,a^{6}\,x^{20} +$
$3\,108\,105\,a^{8}\,x^{20} - 542\,928\,122\,880\,a\,x^{21} + 35\,274\,919\,680\,a^{3}\,x^{21} - 430\,990\,560\,a^{5}\,x^{21} +$
$1\,184\,040\,a^{7}\,x^{21} - 24\,678\,551\,040\,x^{22} + 4\,810\,216\,320\,a^{2}\,x^{22} - 97\,952\,400\,a^{4}\,x^{22} + 376\,740\,a^{6}\,x^{22} +$
$418\,279\,680\,a\,x^{23} - 17\,035\,200\,a^{3}\,x^{23} + 98\,280\,a^{5}\,x^{23} + 17\,428\,320\,x^{24} - 2\,129\,400\,a^{2}\,x^{24} +$
$20\,475\,a^{4}\,x^{24} - 170\,352\,a\,x^{25} + 3276\,a^{3}\,x^{25} - 6552\,x^{26} + 378\,a^{2}\,x^{26} + 28\,a\,x^{27} + x^{28}$

In[∘]:= `Expand[ (a + x)^28 ] (* this expansion renders 29-Terms *)`

Out[∘]=
$a^{28} + 28\,a^{27}\,x + 378\,a^{26}\,x^{2} + 3276\,a^{25}\,x^{3} + 20\,475\,a^{24}\,x^{4} + 98\,280\,a^{23}\,x^{5} + 376\,740\,a^{22}\,x^{6} +$
$1\,184\,040\,a^{21}\,x^{7} + 3\,108\,105\,a^{20}\,x^{8} + 6\,906\,900\,a^{19}\,x^{9} + 13\,123\,110\,a^{18}\,x^{10} + 21\,474\,180\,a^{17}\,x^{11} +$
$30\,421\,755\,a^{16}\,x^{12} + 37\,442\,160\,a^{15}\,x^{13} + 40\,116\,600\,a^{14}\,x^{14} + 37\,442\,160\,a^{13}\,x^{15} + 30\,421\,755\,a^{12}\,x^{16} +$
$21\,474\,180\,a^{11}\,x^{17} + 13\,123\,110\,a^{10}\,x^{18} + 6\,906\,900\,a^{9}\,x^{19} + 3\,108\,105\,a^{8}\,x^{20} + 1\,184\,040\,a^{7}\,x^{21} +$
$376\,740\,a^{6}\,x^{22} + 98\,280\,a^{5}\,x^{23} + 20\,475\,a^{4}\,x^{24} + 3276\,a^{3}\,x^{25} + 378\,a^{2}\,x^{26} + 28\,a\,x^{27} + x^{28}$