Profiles & Perspectives of *p-Binomial Sequences*

From: J B Barksdale, Jr || Composition: Notes / Lemmas / Theorems
09/28/2018

Article 0: Introduction. The *Binomial Theorem* which describes the termwise expansion of a power of binomial expressions has occupied a central, fundamental role in mathematics since (when in 1676) Issac Newton communicated that result to H. Oldenburg via a letter. The *profile description* of that result could be as follows:

Given the sequence of power functions, $\{x^j\}$, *each initial subsequence*, $\{x^k\}_{k=0}^{k=n}$, *of that polynomial sequence satisfies the following identity:*

(0.1) $$(x+y)^n \;=\; \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

for each integer n $\geq$ *0.*

Later (in the 1700's), Vandermonde noted that the *descending factorial powers* defined by the following product of *n-decreasing-factors in x,*

(0.2) $$x^{(n)} \;=\; x(x-1)(x-2)\ldots(x-n+1),$$

satisfied *Vondermonde's analog* of the binomial theorem.

(0.3) $$(x+y)^{(n)} \;=\; \sum_{k=0}^{n} \binom{n}{k} x^{(n-k)} y^{(k)} .$$

Then, in the 1900's, *N.E. Norlund's analog* emerged (as displayed below).

(0.4) $$(x+y)^{[n]} \;=\; \sum_{k=0}^{n} \binom{n}{k} x^{[n-k]} y^{[k]}$$, where the *n-increasing factors in x,*

(0.5) $$x^{[n]} \;=\; x(x+1)(x+2)\ldots(x+n-1) ,$$

define the *ascending factorial powers of x.*

During the 1820's, *Niels Henrik Abel (1802-1829),* offered the sequence,

(0.6) $$p_n(x) \;=\; x(x-an)^{n-1} ,$$

which constitutes a *sequence of polynomials,* $\{ p_j(x) \}$, such that

(0.7) $$p_n(x+y) \;=\; \sum_{k=0}^{n} \binom{n}{k} p_{n-k}(x)\, p_k(y) .$$

Hence, since the announcement (in 1676) of Newton's *binomial theorem,* continued discoveries of additional *polynomial sequences* which *satisfied the symbolic profile* of his *Binomial Theorem formula* periodically emerged. Mathematicians have proceeded to discover other such *binomial type polynomial sequences.* Notably, in the early 1970's, publications by C.-G. Rota, C. Berge and G. Markowsky *completely characterized delta operators and sequences in polynomial rings* which satisfy the *binomial expansion identity profile* appearing as presented in *Item (0.7);* the author of this manuscript has *adopted* the name reference *p-Binomial Expansion Identity* for that formula, since it symbolically expresses the same relational profile for a *completely characterized class* of *polynomials* which (of course) includes the *power functions* and all previously discovered examples of polynomial sequences that satisfy *Item (0.7).* As can be observed from the above displayed illustrations of *p-binomial sequences,* each such modeled sequence of polynomials satisfies the following description of such sequences.

(0.8) *(i)* $\mathbb{B} = \{p_k \mid \textit{deg}(p_k) = k,\, p_0(x) = 1\}$; and

(ii) $p_n \in \mathbb{B} \implies p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_{n-k}(x)\, p_k(y)$.

Hence, it should be noted that a *p-binomial sequence* is a *basis* for the vector space $\mathbb{P}^\infty$ of polynomials over the real number field $\mathbb{R}$. The remaining articles of this composition are devoted to the development and presentation of this composition's theme title: *Profiles and Perspectives of p-Binomial Sequences.*

Article 1: Polynomial Expansion Systems. Imagine the below described *basis* $\mathbb{G} \subset \mathbb{P}^\infty$ for the vector space of polynomials $\mathbb{P}^\infty$ over the real number field $\mathbb{R}$. Hence, consider a *simple basis* of polynomials declared by *Item (1.1).*

(1.1) $\mathbb{G} = \{g_k(x) \mid g_0(x) \equiv 1,\, \deg(g_k) = k\}$ *(simple basis: one g_k of each degree; $g_0 \equiv 1$)* .

Now imagine a *unit degree decreasing (udd) Linear Operator L* defined on $\mathbb{P}^\infty$. Next, consider the notion of constructing an *Expansion System* $\langle L, \mathbb{G}\rangle$ such that for each $f \in \mathbb{P}^\infty$, $f(x) = \sum_{k=0}^{\infty} \frac{L^k f(0)}{k!} g_k(x)$.

Note that no infinite convergence applies here since f denotes a polynomial of *unspecified, finite degree.* Hence, for $f \in \mathbb{P}^\infty$: $\deg(f) = n$, $c_n \neq 0$ and $m > n \Rightarrow c_m = 0$ ($c_k =$ *coefficient of* x^k) .

Article 2: Concept and Structure of Expansion Systems. The following developments establish results which define the concept and the features of *Polynomial Expansion Systems* $\langle L, \mathbb{G}\rangle$ as mentioned in *Article 1.*

Theorem 2.01. Consider a pair $\langle L, \mathbb{G}\rangle$ consisting of a *unit degree decreasing (udd) linear operator (L)* and *a simple basis* $(\mathbb{G})$.
Then: $\forall\, f \in \mathbb{P}^\infty$,

(2.1) $f(x) = \sum_{k=0}^{\infty} \frac{L^k f(0)}{k!} g_k(x) \iff L^k g_j(0) = \delta_j^k\, k!$, where $\delta_j^k = \begin{cases} 1, & k = j \\ 0, & k \neq j \end{cases}$

denotes the *Kronecker Delta*.

Proof. First, *suppose* $f(x) = \sum_{k=0}^{\infty} \frac{L^k f(0)}{k!} g_k(x)$, $\forall f \in \mathbb{P}^\infty$. Hence, $g_j \in \mathbb{P}^\infty$

$$\Longrightarrow g_j(x) = \sum_{k=0}^{\infty} \frac{L^k g_j(0)}{k!} g_k(x) \implies \frac{L^k g_j(0)}{k!} = \delta_j^k$$, since $\mathbb{G}$ is a *basis.*

Thus, $L^k g_j(0) = \delta_j^k \, k!$, as claimed. Note that $g_k^{\#}(f) = L^k f(0)$ defines *linear functionals* for $f \in \mathbb{P}^\infty$; hence, the linear functionals $\mathbb{G}^{\#} = \{ g_k^{\#} \}$ belong to the *algebraic conjugate space* $(\mathbb{P}^\infty)^{\#}$ of $\mathbb{P}^\infty$. Moreover, the basis vectors $\mathbb{G}$ *and companion functionals* $\mathbb{G}^{\#}$ constitute a *bi-orthognal companion pair* since $g_k^{\#}(g_j) = L^k g_j(0) = \delta_j^k \, k!$.

Conversely, *now suppose* that $L^k g_j(0) = \delta_j^k \, k!$ for $g_j \in \mathbb{G}$. Then, $f \in \mathbb{P}^\infty \implies \exists \, c_j$ such that

$$f(x) = \sum_{j=0}^{\infty} c_j \, g_j(x) \implies L^k f(0) = \sum_{j=0}^{\infty} c_j \, L^k g_j(0) = \sum_{j=0}^{\infty} c_j \, \delta_j^k \, k! \implies L^k f(0) = c_k k!$$

$$\Longrightarrow c_k = \frac{L^k f(0)}{k!} \text{ . Therefore: } f(x) = \sum_{k=0}^{\infty} \frac{L^k f(0)}{k!} g_k(x). \quad \square$$

<u>COROLLARY T2.01</u>: Consistency of the preceding results renders that $Lg_0 = 0$, where $\langle L , \mathbb{G} \rangle$ denotes an *Expansion System.*

Proof. Since L is a *udd Linear Operator*, note that $(Lg_0) \in \mathbb{P}^0$. Now, suppose that $\langle L , \mathbb{G} \rangle$ constitutes an expansion system. Then, application of *Theorem 2.01* yields

$$(Lg_0)(x) = \sum_{k=0}^{0} \frac{L^k (L^1 g_0)(0)}{k!} g_k(x) = \frac{(L^0 L^1) g_0(0))}{0!} g_0(x) = \frac{(L^1 g_0)(0)}{0!} g_0(x)$$

$\Longrightarrow \frac{L^1 g_0(0)}{0!} g_0(x) = \delta_0^1 \cdot g_0(x) = 0 \cdot g_0(x) = 0$. Hence, *if* $\langle L , \mathbb{G} \rangle$ constitutes an *expansion system,* then *Theorem 2.01* implies (as a *necessary condition*) that $Lg_0 = 0$. $\square$

Based on the content of *Theorem 2.01,* clearly certain, specific *companion* features must be intrinsic to the *udd linear operator L* and the *simple basis* $\mathbb{G}$ of a *polynomial expansion system* component pair $\langle L , \mathbb{G} \rangle$. Such certain features can, in fact, be recognized by observing the familiar pair $\langle D, \{x^n\} \rangle$, where $L = D$ is the *derivative operator* and $\mathbb{G} = \{x^n\}$ is the family of *power function polynomials.* Then, of course, a *function expression equivalent to a polynomial* $f \in \mathbb{P}^\infty$ can be *expanded into its Maclaurin Series of power functions:* $f(x) = \sum_{k=0}^{\infty} \frac{D^k f(0)}{k!} x^k$. A review of the *certain features* which support and establish this *familiar model* $\langle D , \{x^n\} \rangle$ include the attributes: *(i)* $x^0 \equiv 1$; *(ii)* $(x^k)\Big|_{x=0} = 0, \; k > 0$; and, *(iii)* $Dx^k = \begin{cases} kx^{k-1}, & k > 0 \\ 0 \, , & k = 0 \end{cases}$. As verified by the following development, the adoption of these certain features is sufficient to *characterize* the structural attributes of a *generalized expansion system* as presented in *Theorem 2.02,* below.

<u>Theorem 2.02</u>. A pair $\langle L , \mathbb{G} \rangle$ consisting of a *(udd) linear operator (L)* and *a simple basis* ($\mathbb{G}$) constitutes an *expansion system* (at zero) *iff :* *(i)* $g_0 \equiv 1$; *(ii)* $g_k(0) = 0, \; k > 0$; and *(iii)* $Lg_k = \begin{cases} kg_{k-1} \, , & k \neq 0 \\ 0 \, , & k = 0 \end{cases}$ are *companion properties* of the $\langle L , \mathbb{G} \rangle$ pair.

Proof. First, suppose the companion pair $\langle L\,,\mathbb{G}\rangle$ constitutes an expansion system (at zero); application of *Theorem 2.01* and the features of $\mathbb{G}$ yield that $L^k g_j(0) \,=\, \delta_j^k\, k!$; further, observe that

(i) $g_0(x) = 1$, since $g_0 \in \mathbb{G}$;

(ii) $g_j(0) = L^0 g_j(0) \,=\delta_j^0\, 0! \,=\, 0\,,\; j > 0$;

Recall that the *Corollary of Theorem 2.01* renders that $Lg_0(x) = 0$ as a necessary condition whenever $\langle L\,,\mathbb{G}\rangle$ is an expansion system. Furthermore, vigilant attention to symbolic detail yields the following equalities.

(2.2) $$L^k(Lg_j(0)) \,=\, L^k((Lg_j)(x))\Big|_{x=0} = L^{k+1}g_j(x)\Big|_{x=0} \,=\, L^{k+1}g_j(0) \,=\, \delta_j^{k+1}\,(k{+}1)!\,.$$

Applying the hypothesis that $\langle L\,,\mathbb{G}\rangle$ constitutes an expansion system (at zero) renders

(iii) $$Lg_j(x) \,=\sum_{k=0}^{\infty}\tfrac{L^k(Lg_j)(0)}{k!}\,g_k(x) \,=\sum_{k=0}^{\infty}\tfrac{L^{k+1}g_j(0)}{k!}\,g_k(x)$$

$$=\,\sum_{k=0}^{\infty}\tfrac{\delta_j^{k+1}\,(k+1)!}{k!}\,g_k(x) \,=\, \tfrac{j!}{(j-1)!}\,g_{j-1}(x) \,=\, j\,g_{j-1}(x)\,.$$

Accordingly, *conditions (i), (ii) and (iii)* are *necessary properties* of an *expansion system* companion pair $\langle L\,,\mathbb{G}\rangle$.

Conversely, now suppose that *properties (i), (ii) and (iii)* are met for a given companion pair $\langle L\,,\mathbb{G}\rangle$. Then note that: *(a)*-- repeated application of *property (iii)* $\Longrightarrow\; L^k g_j(x) \,=\, \tfrac{j!}{(j-k)!}\,g_{j-k}(x)$ *;* hence, *property (ii)* renders implications: $k{-}j < 0 \Longrightarrow L^k g_j(0) = \tfrac{j!}{(j-k)!}\,g_{j-k}(0) \,=\, 0 \Longrightarrow L^k g_j(0) \,=\, 0$; *(b)*--also observe that *properties (i), (iii)* yield the implication: $k{-}j = 0 \Longrightarrow L^k g_j(0) = \tfrac{k!}{0!}\,g_0(0) = k!$; *(c)--properties (i), (iii)* also produce: $k - j > 0 \;\Longrightarrow\; L^k g_j(0) = L^{k-j}L^j g_j(0) = L^{k-j}\,j!\,g_0(0) = 0$. Consequently, summarizing *observations (a), (b) and (c),* renders

(2.3) $$L^k g_j(0) \;=\; \begin{cases} 0 & ,\; k{-}j < 0 \\ k! & ,\; k{-}j = 0 \\ 0 & ,\; k{-}j > 0 \end{cases} \qquad = \; k!\,\delta_j^k\,.$$

Thus, a *companion pair* $\langle L\,,\mathbb{G}\rangle$ *endowed with properties (i), (ii)* and *(iii)* constitutes an *expansion system* as a consequence of *Theorem 2.01.* ☐

Reviewing the first few lines appearing in the proof of *Theorem 2.02*, it is observed that *necessary conditions* for an expansion system companion pair $\langle L\,,\mathbb{G}\rangle$ are that: *(i)* $g_0(x) = 1$, since $g_0 \in \mathbb{G}$; and *(ii)* $g_j(0) = \, 0\,,\; j > 0$; further, *COROLLARY T2.01* established that $Lg_0(x) = 0$. Accordingly, a *simple basis* denoted by the *symbol* $\mathbb{G}$ shall be modified to *include conditions (i) and (ii),* above, and (as a new family) called a *normal family basis* $\mathbb{G}$. Clearly, *all preceding proofs and results* remain unaffected by interchanging the concept references *"simple basis* $\mathbb{G}$*"* and *"normal family basis* $\mathbb{G}$*"* in the statements and developments appearing in these preceding articles.

Hence, in this composition, future references to the *symbol* $\mathbb{G}$ shall refer to the below listed (2.4) *modified definition.*

(2.4) $\mathbb{G} = \{g_k(x) \,\big|\, g_0(x) \equiv 1,\ \deg(g_k) = k\,,\ g_k(0) = 0\ \ (k > 0)\,\}$ *(normal family basis)*

Hence, this modified definition of $\mathbb{G}$ with *condition (ii) attached,* shall now have the reference name *normal family basis.*

Theorem 2.03. If $f(x) = \sum_{k=0}^{\infty} \frac{L^k f(0)}{k!}\, g_k(x)\,,\ \forall\, f \in \mathbb{P}^\infty$, then L *and* $\mathbb{G} = \{g_k\}$ constitute a *unique (each to the other) companion pair* $\langle L\,,\mathbb{G}\rangle$ as an *expansion system* for $\mathbb{P}^\infty$.

Proof. First, *given L,* suppose there exists *two companion normal family bases,* $\mathbb{G}$ *and* $\mathbb{H}$=$\{h_j\}$ for the *operator* L. Thus, $g_k \in \mathbb{G} \subset \mathbb{P}^\infty \implies g_k(x) = \sum_{j=0}^{\infty} \frac{L^j g_k(0)}{j!}\, h_j(x) \ = \sum_{j=0}^{\infty} \delta_k^j\, h_j(x) \ = \ h_k(x).$
Thus, for *each index k:* $h_k = g_k \implies \mathbb{H} = \mathbb{G}$, and so $\mathbb{G}$ is the *unique companion* for L.
Conversely, now suppose that *given a normal family bases* $\mathbb{G}$ there exist *companion operators L and M.* Hence, for *each index k:* $Lg_k \ = \ k\, g_{k-1} \ = \ M g_k$, for *both systems* $\langle L\,,\mathbb{G}\rangle$ *and* $\langle M\,,\mathbb{G}\rangle$.
Consequently, $\forall\, f \in \mathbb{P}^\infty\,,\ \exists\, c_k \in \mathbb{R}$ such that: $f(x) \ = \sum_{k=0}^{\infty} c_k\, g_k(x) \implies Lf(x) \ = \sum_{k=0}^{\infty} c_k\, Lg_k(x)$
$= \sum_{k=0}^{\infty} c_k\, M g_k(x) \ = \ M f(x).$ Such last implication establishes the *unique companion character of* L for the given *normal family basis* $\mathbb{G}$ *;* therefore, the proof of *Theorem 2.03* is complete. ☐

Theorem 2.04. Given *either* component L, or $\mathbb{G}$, of an expansion system pair $\langle L\,,\mathbb{G}\rangle$, then, *there exists an unique companion*, $\mathbb{G}$, or L, which completes an expansion system pair $\langle L\,,\mathbb{G}\rangle$.

Proof. First, *suppose* that a *linear (udd) operator* L on $\mathbb{P}^\infty$ is given. Hence, a *normal family basis* is needed to pair with L. Note that both $\mathbb{X} = \{x^k\}_{k=0}^{\infty}$ and $L(\mathbb{X}) = \{Lx^k\}_{k=1}^{\infty}$ are actually *bases* for $\mathbb{P}^\infty$.
Consider the following *inductive definition* used to construct a *normal family basis* denoted by $\mathbb{G}$.
Hence, construct *linear combinations* from *bases* $\mathbb{X}$ and $L(\mathbb{X})$ as described in (2.4).

(2.4) $g_0(x) = 1$, declared by definition. Now, since $L(\mathbb{X})$ is a *basis,* $\exists\, c_{01} \in \mathbb{R}$ such that,
$g_0(x) = c_{01} L[x^1] \ = \ L[c_{01} x^1]$;
♦define: $g_1(x) = c_{01} x^1$; Again, since $L(\mathbb{X})$ is a *basis*, then
$2g_1(x) = c_{11} Lx^1 + c_{12} Lx^2 \ = L[c_{11} x + c_{12} x^2]$;
♦define: $g_2(x) = c_{11} x + c_{12} x^2$;
⋮
$n g_{n-1}(x) \ = \ c_{n-1,1} Lx^1 + c_{n-1,2} Lx^2 + \ldots + c_{\,n-1,n} Lx^n = L\left[\sum_{k=1}^{n} c_{\,n-1,k}\, x^k\right]$;
♦define: $g_n(x) \ = \sum_{k=1}^{n} c_{\,n-1,k}\, x^k$.

Verification that the $\mathbb{G} = \{g_n\}$ from (2.4) is a companion *normal family basis* is confirmed by noting that such construction satisfies the properties: *(i)* $g_0(x) = 1$; *(ii)* $g_k(0) = 0,\ k > 0$; and, of course,

(iii) $Lg_n = ng_{n-1}$. Thus, by *Theorem 2.02*, for a *given* L, there exists a *unique normal family basis companion* $\mathbb{G}$ such that $\langle L\,, \mathbb{G}\rangle$ constitutes an expansion system pair.
Conversely, *suppose* that a *normal family basis* $\mathbb{G}$ is given. Now, proceed to *define* a companion *(udd) Operator L* on $\mathbb{P}^\infty$ by asserting: *(a)* $Lg_0 = 0$; *(b)* $Lg_n = ng_{n-1}$ *(c)* L *is linear*. Consequently, since $\mathbb{G}$ is a basis, then L is a *(udd) Linear Operator* defined on $\mathbb{P}^\infty$ and which satisfies the prescribed properties. Therefore, again, by *Theorem 2.02*, for a *given* $\mathbb{G}$, there exists a *unique (udd) Operator L* such that $\langle L\,, \mathbb{G}\rangle$ constitutes an expansion system pair. □

Article 3: p-Binomial Sequences and Shift Operators. As presented in the introductory article of this composition, publications of noted mathematicians during the 1970's created fundamental results which provided a framework which rendered structural relationships that completely characterized sequences of polynomials that satisfied the *p-Binomial Identity* referenced in *Article 0*,
$p_n(x+y) = \sum_{k=0}^{n}\binom{n}{k}\, p_{n-k}(x)\, p_k(y)$. Some of those concepts are hereby presented in this article.

(3.1a) DEFINITION. An *Operator* $\mathbb{P}^\infty \xrightarrow{T_a} \mathbb{P}^\infty$ defined by: $T_a f(x) = f(a+x)$, for each $a \in \mathbb{R}$, is referenced as a *Translation* or *Shift* operator on the vector space of polynomials $\mathbb{P}^\infty$ over the *real number field* $\mathbb{R}$.

(3.1b) DEFINITION. An *Operator* $\mathbb{P}^\infty \xrightarrow{L} \mathbb{P}^\infty$ is called *Shift Invariant* or T_a*-Invariant* whenever: $LT_a = T_a L$, $\forall\, a \in \mathbb{R}$.

(3.2) DEFINITION. A *p-Binomial Sequence,* $S_p = \{p_k\} \subset \mathbb{P}^\infty$, is defined as a *sequence of polynomials* each term of which satisfies the *p-Binomial Identity*.

Establishment of the following developments are achieved via application of the theorems and results appearing in the here presented preceding articles.

Theorem 3.01. Given an expansion system $\langle L\,, \mathbb{G}\rangle$, then,
$\mathbb{G}$ constitutes a *p-Binomial Sequence* $\iff$ *L* is *Shift Invariant* (T_a*-Invariant*).

Proof. First, *suppose* that $\mathbb{G}$ constitutes a *p-Binomial Sequence*. Then,

$$g_n \in \mathbb{G} \implies (LT_a)g_n(x) = Lg_n(a+x) = L\left[\sum_{k=0}^{n}\tbinom{n}{k}\, g_{n-k}(a)\, g_k(x)\right]$$

$$= \sum_{k=0}^{n}\tbinom{n}{k}\, g_{n-k}(a)\, Lg_k(x) = \tbinom{n}{0} g_n(a) Lg_0(x) + \sum_{k=1}^{n}\tbinom{n}{k}\, g_{n-k}(a)\, Lg_k(x)$$

$$= \sum_{k=1}^{n}\tbinom{n}{k}\, g_{n-k}(a)\, kg_{k-1}(x) = \sum_{k=1}^{n}\tbinom{n}{k} k\, g_{n-k}(a)\, g_{k-1}(x) = \sum_{k=1}^{n} n\tbinom{n-1}{k-1}\, g_{n-k}(a)\, g_{k-1}(x)$$

$$= n\sum_{k=0}^{(n-1)}\binom{(n-1)}{k}\, g_{(n-1)-k}(a)\, g_k(x) = n\, g_{n-1}(a+x) = T_a[ng_{n-1}(x)] = T_a[Lg_n(x)]$$

$$= (T_a L)g_n(x) \implies L \text{ is } T_a\text{-Invariant on } \mathbb{G} \implies L \text{ is also } T_a\text{-Invariant on } \mathbb{P}^\infty,$$

since $\mathbb{G}$ also serves as a *basis* for $\mathbb{P}^\infty$.

Conversely, *supposing* that L is T_a*-Invariant* $\implies Lg_n(a{+}x) = (LT_a)g_n(x) \;=\; (T_aL)g_n(x)$ $\;=\; T_a[n\,g_{n-1}(x)] \;= n\,g_{n-1}(a{+}x)$. Thus, $L^k g_n(a{+}x) = \binom{n}{k}k!\,g_{n-k}(a{+}x)$, by repeated application of L. Additionally, since $\mathbb{G}$ is a basis, then: $g_n(a+x) \;= \sum\limits_{j=0}^{n} c_j\; g_j(x)$, and therefore applying L^k yields,

$L^k g_n(a+x) \;= \sum\limits_{j=0}^{n} c_j\, L^k g_j(x) \implies \binom{n}{k}k!\,g_{n-k}(a{+}x) \;=\; \sum\limits_{j=0}^{n} c_j\, L^k\; g_j(x)$ *; evaluating* at $x = 0$ renders,

$\binom{n}{k}k!\,g_{n-k}(a) \;=\; \sum\limits_{j=0}^{n} c_j\, L^k\; g_j(0) \implies \binom{n}{k}k!\,g_{n-k}(a) \;=\; \sum\limits_{j=0}^{n} c_j\, k!\,\delta_j^k \;=\; c_k\, k!$ *(by Theorem 2.01)*

$\implies c_k \;=\; \binom{n}{k}\, g_{n-k}(a)$, applying that $\langle L\,,\mathbb{G}\rangle$ is an expansion system. Consequently, g_n satisfies the *p-Binomial Identity* $g_n(a+x) \;= \sum\limits_{k=0}^{n} \binom{n}{k}\; g_{n-k}(a)\; g_k(x)$. Therefore, $\mathbb{G}$ is a *p-Binomial Sequence.* □

> **<u>Theorem 3.02</u>.** Every *p-Binomial Sequence* $\mathbb{G} \subset \mathbb{P}^\infty$ is a *companion, normal family basis* for a *unique* T_a*-Invariant (udd) Linear Operator L* such that the two such components constitute an *unique (each to the other) expansion system pair* $\langle L\,,\mathbb{G}\rangle$.

<u>Proof.</u> Given that $\mathbb{G}$ is a *p-Binomial Sequence,* then

$g_n \in \mathbb{G}$ *and* $0 \in \mathbb{R} \implies g_n(x) \;=\; g_n(0{+}x) \;=\; \sum\limits_{k=0}^{n} \binom{n}{k}\; g_{n-k}(0)\; g_k(x) \implies g_n(0) = 0,\; g_0(0) = 1$,

since $\mathbb{G}$ is a *basis;* thus, $g_k(0) = 0,\; k > 0\,,\; g_0(x) \equiv 1$ implies that $\mathbb{G}$ is a *normal family basis.* Hence, application of *Theorem 2.04* asserts the existence of a *unique, (udd) operator L* such that $\langle L\,,\mathbb{G}\rangle$ constitutes an expansion system pair. Furthermore, as a consequence of *Theorem 3.01*, the operator L must be T_a*-Invariant.* □

<u>SUB-ARTICLE S3.1</u>. Prior to proceeding with further theorems and results regarding the *characterization* and *construction* of *p-Binomial Sequences,* certain notions and concepts of structure are required in order to devise such developments. Hence, the below mentioned *notes & tools* shall now be introduced:

<u>(S3.1a)</u>: Imagine a *square*, *Upper-Triangular matrix* $[M] = [m_{ij}]\,,\; 0 \le i{,}j \;< \infty$;
Now, suppose that *all element values of each super-diagonal* equal to the *leading element value* $m_{0,n}$ which belongs to *row number zero (0)* of that super-diagonal. Hence, a *description* of such *constant*, super-diagonal element values is: $m_{r,n+r} = m_{0,n}$, where $r \ge 0, n \ge 1$. Therefore, $a < b$ $\implies m_{a,b} = m_{0,b-a}$; $a > b \implies m_{ab} = 0$, $[M]$ is *UT.*

<u>(S3.1b)</u>: Note derivatives for *canonical basis* $\mathbb{B} = \{\frac{x^j}{j!}\}$: $D^k\left[\frac{x^j}{j!}\right] \;=\; \boxed{\frac{x^{j-k}}{(j-k)!}\,,\; k \le j}$.

Consequently: $f \in \mathbb{P}^\infty \implies f^{(k)}(x) =\; D^k\left[\sum\limits_{j=0}^{\infty}\frac{f^{(j)}(0)}{j!}\,x^j\right] = \sum\limits_{j=0}^{\infty} f^{(j)}(0)D^k\left[\frac{x^j}{j!}\right]$

$$= \sum_{j=0}^{k-1} f^{(j)}(0)D^k\left[\tfrac{x^j}{j!}\right] + \sum_{j=k}^{\infty} f^{(j)}(0)D^k\left[\tfrac{x^j}{j!}\right] \;=\; \boxed{\sum_{j=k}^{\infty} f^{(j)}(0)\; \tfrac{x^{j-k}}{(j-k)!}}\;.$$

(S3.1c): The *matrix* $[T_a] = [\tau_{kj}]$ for a *Shift Operator* $\mathbb{P}^\infty \xrightarrow{T_a} \mathbb{P}^\infty$ *w.r.t. the basis* $\mathbb{B}$, *above, is described by: (i) Upper-Triangular ; and, (ii)* $\tau_{kj} = \frac{a^{j-k}}{(j-k)!}$, $j \leq k$.

<u>(S3.1d)</u>:

$$\sum_{k=a}^{b} S_k = \sum_{k=0}^{b-a} S_{k+a} \qquad \textit{(shift index and limits)} ;$$

$$\sum_{k=a}^{b} S_k = \sum_{k=a}^{b} S_{(a+b)-k} \quad \textit{(reverse order of summation)} ;$$

$$\sum_{k=a}^{b} S_k = \sum_{k=0}^{b-a} S_{b-k} \qquad \textit{(shift index and limits, reverse sum order).}$$

<u>(S3.1e)</u>: *Products of UT-matrices are also UT-matrices.* Hence, $[L][T_a]$ and $[T_a][L]$ are *UT-matrices* if L is T_a*-invariant,* since $LT_a = T_a L$ implies that $[L][T_a] = [T_a][L]$.

Hence: $[\lambda_{ik}][\tau_{kj}] = \left[\sum_{k=i+1}^{j} \lambda_{ik}\tau_{kj}\right] = \left[\sum_{k=i+1}^{j} \lambda_{ik}\frac{a^{j-k}}{(j-k)!}\right]$;

here, for *lower sum limit:* $i \geq k \Rightarrow \lambda_{ik} = 0$; hence, start $k = i+1 > i$; also, note: $k > j \Rightarrow \tau_{kj} = 0$; hence, keep $k \leq j =$ *upper sum limit.*

Now, similarly: $[\tau_{ik}][\lambda_{kj}] = \left[\sum_{k=i}^{j-1} \tau_{ik}\lambda_{kj}\right] = \left[\sum_{k=i}^{j-1} \frac{a^{k-i}}{(k-i)!}\lambda_{kj}\right]$; again, note that *sum limits* were chosen appropriately as such was for the prior sums. Consequently, *given that*

L *is* T_a*-invariant* $\Longrightarrow [\lambda_{ik}][\tau_{kj}] = [\lambda_{ik}][\tau_{kj}] \Longrightarrow \sum_{k=i+1}^{j} \lambda_{ik}\frac{a^{j-k}}{(j-k)!} = \sum_{k=i}^{j-1} \frac{a^{k-i}}{(k-i)!}\lambda_{kj}$;

observe that the *exponent values* $(j-k)$ *and* $(k-i)$ *range over the same consecutive integer sequence;* and, note that *first sum exponent values* occur in *reverse order* of those *same values* in the *second sum.* Hence, the below implemented: *index shift and sum-limits revision* shall *illustrate* a comparison of these *two, equal descriptions* of the *(i,j)-position element of these equal matrix products.* Hence, applying the tools appearing in (S3.1d) renders,

$$\sum_{k=i+1}^{j} \lambda_{ik}\tau_{kj} = \sum_{k=0}^{j-i-1} \lambda_{i,j-k}\tau_{k-j,j} = \sum_{k=0}^{j-i-1} \lambda_{i,j-k}\frac{a^k}{k!} ; \text{ similarly,}$$

$$\sum_{k=i}^{j-1} \tau_{ik}\lambda_{kj} = \sum_{k=0}^{j-i-1} \tau_{i,k+i}\lambda_{k+i,j} = \sum_{k=0}^{j-i-1} \frac{a^k}{k!}\lambda_{k+i,j} .$$

Observe that the two *linear combinations of the same basis elements* $\{\frac{a^k}{k!}\}$, yield equality of the *scalars:* $\lambda_{i,j-k} = \lambda_{k+i,j}$. Appeal to the notes appearing in (S3.1a) presents: $i = 0,\ j = n+k \Longrightarrow \boxed{\lambda_{0,n} = \lambda_{k,n+k}}$.

<u>(S3.1f)</u>: CONSEQUENTLY, in view of the notes at (S3.1a):
The UT-matrix $[L]$ $(w.r.t.\ \mathbb{B} = \{\frac{x^k}{k!}\})$ *of a (udd)* T_a*-invariant*

Operator L on $\mathbb{P}^\infty$ exhibits equal, constant-value elements on its super-diagonals.

(e.g. , $\lambda_{0,1} = \lambda_{1,2} = \lambda_{2,3} = \lambda_{3,4} = \lambda_{4,5} = \lambda_{5,6} \ldots$
$\lambda_{0,n} = \lambda_{k,n+k}$; $n = 1$; $k = 0, 1, 2, 3, 4, 5, \ldots$

$\lambda_{0,2} = \lambda_{1,3} = \lambda_{2,4} = \lambda_{3,5} = \lambda_{4,6} = \lambda_{5,7} \ldots$
$\lambda_{0,n} = \lambda_{k,n+k}$; $n = 2$; $k = 1, 2, 3, 4, 5, \ldots$

$\lambda_{0,4} = \lambda_{1,5} = \lambda_{2,6} = \lambda_{3,7} = \lambda_{4,8} = \lambda_{5,9} \ldots$,
$\lambda_{0,n} = \lambda_{k,n+k}$; $n = 4$; $k = 0, 1, 2, 3, 4,$ ...... etc) .

(S3.1g): Suppose that a *given, (udd) Operator L on $\mathbb{P}^\infty$ is T_a-Invariant.* Then, the prior note of (S3.1f) applies; hence, the *UT-matrix* $[L]_{\mathbb{B}}$ exhibits *equal-value elements* on its *super-diagonals* (as above illustrated). Furthermore, the *usual Maclaurin expansion* of a polynomial $f \in \mathbb{P}^\infty$ appears as follows: $f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} x^j = \sum_{j=0}^{\infty} f^{(j)}(0) \frac{x^j}{j!}$; hence,

$$Lf(x) = \sum_{j=0}^{\infty} f^{(j)}(0)\, L\left[\tfrac{x^j}{j!}\right] = f^{(0)}(0) L\left[\tfrac{x^0}{0!}\right] + \sum_{j=1}^{\infty} f^{(j)}(0)\, L\left[\tfrac{x^j}{j!}\right] = \sum_{j=1}^{\infty} f^{(j)}(0) \sum_{i=0}^{j-1} \lambda_{ij} \left[\tfrac{x^i}{i!}\right]$$

$$= f^{(1)}(0) \sum_{i=0}^{0} \lambda_{i1} \left[\tfrac{x^i}{i!}\right] + f^{(2)}(0) \sum_{i=0}^{1} \lambda_{i2} \left[\tfrac{x^i}{i!}\right] + f^{(3)}(0) \sum_{i=0}^{2} \lambda_{i3} \left[\tfrac{x^i}{i!}\right] + \ldots + f^{(n)}(0) \sum_{i=0}^{n-1} \lambda_{in} \left[\tfrac{x^i}{i!}\right] + \cdots .$$

Now, from *this last line of sums, collect ALL SUM COEFFICIENTS for i=0;* then, *collect all SUM coefficients* for $i = 1$; then, *for i=2,...etc.;* that is, *collect and associate* the *coefficients of like basis elements* $\frac{x^i}{i!}$ *from each sum* and *for each* $i = 0, 1, 2, \ldots$. Note such results as below illustrated in the subsequent *display (S3.1gg).*

(S3.1gg)

$$Lf(x) =$$
$$(\lambda_{01} f^{(1)}(0) + \lambda_{02} f^{(2)}(0) + \lambda_{03} f^{(3)}(0) + \lambda_{04} f^{(4)}(0) + \lambda_{05} f^{(5)}(0) + \ldots) \tfrac{x^0}{0!}$$
$$+ (\lambda_{12} f^{(2)}(0) + \lambda_{13} f^{(3)}(0) + \lambda_{14} f^{(4)}(0) + \lambda_{15} f^{(5)}(0) + \ldots) \tfrac{x^1}{1!}$$
$$+ (\lambda_{23} f^{(3)}(0) + \lambda_{24} f^{(4)}(0) + \lambda_{25} f^{(5)}(0) + \lambda_{26} f^{(6)}(0) + \ldots) \tfrac{x^2}{2!}$$
$$+ (\lambda_{34} f^{(4)}(0) + \lambda_{35} f^{(5)}(0) + \lambda_{36} f^{(6)}(0) + \lambda_{37} f^{(7)}(0) + \ldots) \tfrac{x^3}{3!}$$
$$+ (\ldots \text{etc.} \ldots) .$$

(NOTE: *Re-organize as indicated by the Colored Flow-Lines*)

Next, *re-organize the associated-terms display (S3.1gg) and re-formulate* $Lf(x)$ *as follows:*

$$Lf(x) = (\lambda_{01} f^{(1)}(0) \tfrac{x^0}{0!} + \lambda_{12} f^{(2)}(0) \tfrac{x^1}{1!} + \lambda_{23} f^{(3)}(0) \tfrac{x^2}{2!} + \lambda_{34} f^{(1)}(0) \tfrac{x^3}{3!} + \ldots)$$
$$+ (\lambda_{02} f^{(2)}(0) \tfrac{x^0}{0!} + \lambda_{13} f^{(3)}(0) \tfrac{x^1}{1!} + \lambda_{24} f^{(4)}(0) \tfrac{x^2}{2!} + \lambda_{45} f^{(5)}(0) \tfrac{x^3}{3!} + \ldots)$$
$$+ (\lambda_{03} f^{(3)}(0) \tfrac{x^0}{0!} + \lambda_{14} f^{(4)}(0) \tfrac{x^1}{1!} + \lambda_{25} f^{(5)}(0) \tfrac{x^2}{2!} + \lambda_{36} f^{(6)}(0) \tfrac{x^3}{3!} + \ldots) + (\ldots \text{etc.} \ldots) .$$

$+$ (... etc. ...) .

<u>(S3.1h)</u>: CONSEQUENTLY, in view of the notes at (S3.1a) and (S3.1b):

$$Lf(x) = \sum_{j=1}^{\infty} \lambda_{j-1,j} f^{(j)}(0) \tfrac{x^{j-1}}{(j-1)!} + \sum_{j=2}^{\infty} \lambda_{j-2,j} f^{(j)}(0) \tfrac{x^{j-2}}{(j-2)!} + ... + \sum_{j=k}^{\infty} \lambda_{j-k,j} f^{(j)}(0) \tfrac{x^{j-k}}{(j-k)!} + ...$$

$$= \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} \lambda_{j-k,j} f^{(j)}(0) \tfrac{x^{j-k}}{(j-k)!} = \sum_{k=1}^{\infty} \lambda_{0,k} \left(\sum_{j=k}^{\infty} f^{(j)}(0) \tfrac{x^{j-k}}{(j-k)!} \right) = \sum_{k=1}^{\infty} \lambda_{0,k} f^{(k)}(x)$$

$$= \sum_{k=1}^{\infty} \lambda_{0,k} D^k f(x) = \left(\sum_{k=1}^{\infty} \lambda_{0,k} D^k \right) f(x) \implies \boxed{ L = \sum_{k=1}^{\infty} \lambda_{0,k} D^k } .$$

Sub-Article note (S3.1h) concludes <u>*Sub-Article (S3.1)*</u> and completes the need to introduce developmental *concepts, tools and utilities* required to devise further theorems and results regarding the *structural attributes* and *construction methods* associated with *polynomial expansion systems* $\langle L , \mathbb{G} \rangle$ and the *allied, unique (each to the other) companion pairs.* Therefore, continued *discussions and developments* of <u>*Article 3: p-Binomial Sequences and Shift Operators*</u> is now re-opened to resume.

> **<u>Theorem 3.03</u>.** A *(udd) linear Operator L on* $\mathbb{P}^{\infty}$ is *shift-invariant (T_a-invariant)* iff L has the *representation:* $L = \sum_{k=1}^{\infty} \lambda_k D^k$, where $\{\lambda_k\} \subset \mathbb{R}$ is a sequence with $\lambda_1 \neq 0$, (given that L is *udd*).

<u>*Proof*</u>. First, *suppose that:* $L = \sum_{k=1}^{\infty} \lambda_k D^k$, $\lambda_1 \neq 0$. Now, note that:

$f \in \mathbb{P}^{\infty}$, $a \in \mathbb{P}^{\infty} \implies (T_a D^k) f(x) = T_a f^{(k)}(x) = f^{(k)}(a+x)$
$= D^k f(a+x) = D^k (T_a f(x)) = (D^k T_a) f(x)$;

consequently, D^k is T_a*-invariant*. Thus, $f \in \mathbb{P}^{\infty}$, $a \in \mathbb{R}$ *implies* that:

$(T_a L) f = \sum_{k=1}^{\infty} \lambda_k (T_a D^k) f = \sum_{k=1}^{\infty} \lambda_k (D^k T_a) f = \sum_{k=1}^{\infty} \lambda_k (D^k (T_a f) = L(T_a f) = (L T_a) f$. Therefore,

$L = \sum_{k=1}^{\infty} \lambda_k D^k$, $\lambda_1 \neq 0 \implies$ *Operator L is T_a-invariant.* Conversely, *suppose that L is T_a-invariant.* Then, the *chained, collective, and refined content* presented by <u>*Sub-Ariticle S3.1*</u> and *concluding* with *sub-article **(S3.1h)*** establishes that: $L T_a = T_a L \implies \exists \{\lambda_k\} \subset \mathbb{R}$, $\lambda_1 \neq 0$ such that $L = \sum_{k=1}^{\infty} \lambda_k D^k$. ☐

COROLLARY T3.03: A *normal family basis* $\mathbb{G}$ is a *p-Binomial Sequence iff* the *unique, companion (udd) Linear Operator* has the form: $L = \sum_{k=1}^{\infty} \lambda_k D^k$, *where:* $\{\lambda_k\} \subset \mathbb{R}$, $\lambda_1 \neq 0$.

Proof. The *chained, collective* contents of *Theorems 3.01 and 3.03* constitute a proof demonstration for this corollary of *Theorem 3.03*. ☐

Article 4: p-Binomial Sequence Modeling Devices. Prior articles of this composition have established the *existential* and *uniqueness properties* associated with *(udd) Linear Operators* L and their *companion normal family Bases* $\mathbb{G}$ to create a system pair $\langle \mathbb{G} , L \rangle$*;* further developments revealed that: *a normal family basis* $\mathbb{G}$ *is a p-Binomial Sequence IFF its companion Operator L is Shift-Invariant*. Regarding the *prior developmental articles*, observe that given *(udd) Linear Operators* $\mathbb{P}^\infty \xrightarrow{L} \mathbb{P}^\infty$, and the *normal family bases* $\mathbb{G} \subset \mathbb{P}^\infty$ were of *unspecified cardinality;* hence: *(i) given* $L = \sum_{k=1}^{\infty} \lambda_k D^k$, $\{\lambda_k\}$ options were: *finite OR infinite non-zero terms; (ii) given* $\mathbb{G} = \{g_k\}$ options were: *finite OR infinite basis*. Consequently, the *cardinality* of neither $\{\lambda_k\}$ nor $\{g_k\}$ appertained to any prior developments of this composition. Accordingly, as an illustration, another *DIRECT COROLLARY of Theorem 3.03 is:*

COROLLARY T3.03B: A *normal family basis* $\mathbb{G} \subset \mathbb{P}^n$ is a *p-Binomial Sequence iff* the *unique, companion (udd) Linear Operator* L has the form: $L = \sum_{k=1}^{n} \lambda_k D^k$, *where:* $\{\lambda_k\}_{k=1}^{n} \subset \mathbb{R}$, $\lambda_1 \neq 0$.

Thus, the *cardinality of a* p*-Binomial Sequence* can be either *a finite sequence* $\{p_k\}_{k=0}^{n}$, *or an infinite sequence* $\{p_k\}_{k=0}^{\infty}$ of polynomials in $\mathbb{P}^\infty$. Hence, *every finite, initial subsequence* ,$\{p_k\}_{k=0}^{n}$ *(serving as a basis for* $\mathbb{P}^n$*), of an infinite p-Binomial Sequence is a finite p-Binomial Sequence.* Advancing forward into Article 4, the fundamental prior *characterizations* and *foundational developments* are applied in order to explore and present *operational devices* and *implementations* which *devise* and *render* actual *p-Binomial Sequence Models.* Such devices include: *(i) polynomial, differential operators; (ii)* p_k *inductive definitions; (iii) indefinite integrals;* and, *(iv) linear systems of equations.* Hence, consider the first of several *lemmas and theorems* which are below presented.

LEMMA L4.01. Given an *integer* $q > 0$ and a *polynomial (differential) operator* L defined by $L = L(D) = \sum_{k=1}^{n} \lambda_k D^k$ $(\lambda_1 \neq 0 \neq \lambda_n)$, then: there exists a *m^{th}-order polynomial operator,* $Q = \sum_{j=0}^{m} \mu_j D^j$, $(\mu_0 \neq 0 \neq \lambda_m)$ such that: *(i)* $m > q$; and , *(ii)* $(LQ)f = Df$ for $f \in \mathbb{P}^{m+1}$. (Note: Viewing the *symbol D* as an *indeterminate,* the below referenced polynomial operators can be viewed as *elements of the Euclidean (Integral Domain) Ring* $\mathbb{E}$ *of polynomials (* $\mathbb{P}^\infty$ *over the real number field* $\mathbb{R}$ *).*

Proof. Let such n^{th} *order Operator L* and *integer* $q > 0$ (as above) be given. Noting the *order of ascending powers* of $L(D)$, steps of the *Division Algorithm* are repeatedly applied as *described, detailed and presented* in Item (4.1), below. Note such *application of the DA* is continued until $\mu_m \neq 0$ is encountered, where $m = \inf(M)$, and $M = \{k \in \mathbb{N} \;\big|\; k \geq \max\{n, q\} \;\wedge\; \mu_k \neq 0\}$. Then,

(4.1) $$\frac{D}{L} = \frac{D}{L(D)} = \frac{D}{\sum_{k=1}^{n} \lambda_k D^k} = \sum_{j=0}^{m} \mu_j D^j + \frac{R(D)}{L(D)} = Q(D) + \frac{R}{L},$$

which implies

(4.2) $$D = LQ + R(D) = LQ + R,$$

where $R(D) = D^{m+2}\sigma(D)$, and σ denotes a polynomial of *degree* $(n-2)$. Note the *remainder, R(D),* does have *minimal order* $(m+2)$. Hence, from the above details, the *Division Algorithm* was executed to construct *the quotient* Q and *the remainder R* so that Q *achieved order* $m = \inf(M)$, and so the remainder R, *consequently achieved order* $(n+m)$, and has *minimal order* $(m+2)$. Hence, via *the application of the Division Algorithm* to the *(polynomial) elements D,* $L \in \mathbb{E}$ (*Euclidean domain* $\mathbb{E}$*),* the ring elements Q *and R* were constructed via the *Division Algorithm* for *Euclidean Rings (Domains).* Such ring (domain) elements are thus related as displayed in *Item (4.2).* Hence,

(4.3) $LQ = (D - R)$, via the *Division Algorithm.*

From *Item (4.3)* and the *unique factorization property* for *Euclidean Integral Domains (ring elements),* the operators $L,Q \in \mathbb{E}$ constitute a *unique pair of factors* as a consequence of the *Division Algorithm* applied to *given Euclidean ring elements* $L,D \in \mathbb{E}$. Hence, from the above details, the *Division Algorithm* was executed until *quotient Q* did achieve *order* m, and so that the *remainder* R would have *minimal order* $(m+2)$. Therefore, $p \in \mathbb{P}^{m+1}$ implies

(4.4) $Dp = (LQ + R)p = (LQ)p$, since $\mathbb{P}^{m+1} \subset \text{Ker}(R)$. □

Observe that as a consequence of *Lemma L4.01,* given: *both an integer* $q > 0$ and n^{th}*-order Operator* $L = \sum_{k=1}^{n} \lambda_k D^k$ $(\lambda_1 \neq 0 \neq \lambda_n)$, then: $\exists$ *an Operator* $Q = \sum_{j=0}^{m} \mu_j D^j$, $(\mu_0 \neq 0 \neq \lambda_m)$, $m > q$, such that $\boxed{f \in \mathbb{P}^{m+1} \implies (PQ)f = Df}$. Therefore, the *consequent equality* of the *boxed implication* can be extended to arbitrary *orders of subspaces:* $\mathbb{P}^m \subset \mathbb{P}^\infty$. Thus, in the cases regarding *infinite (non-zero) coefficient sequences:* $L = \sum_{k=1}^{\infty} \lambda_k D^k$, $\{\lambda_k\}_{k=1}^{\infty}$, then *initial, finite subseries of the Operator* L could be implemented in order to create *instances of application for Lemma L4.01;* such application instances would then render corresponding *companion operators* Q and *subspaces* $\mathbb{P}^m$ so that utility of the implication: $f \in \mathbb{P}^{m+1} \implies (PQ)f = Df$, could be applied to *arbitrary m-orders of subspaces* $\mathbb{P}^m$.

LEMMA L4.02. Two Linear Operators *L, Q* *agree on* $\mathbb{P}^{\infty}$ *iff* L , Q *agree on* every *finite subspace* $\mathbb{P}^{n} \subset \mathbb{P}^{\infty} = \bigcup_{m=0}^{\infty} \mathbb{P}^{m}$.

Proof. Let L , Q denote two, linear operators defined on $\mathbb{P}^{\infty}$. First, *suppose* that for *each finite* subspace $\mathbb{P}^{m}$: $f \in \mathbb{P}^{m} \Longrightarrow \; Lf = Qf$; then: $f \in \mathbb{P}^{\infty} = \bigcup_{m=0}^{\infty} \mathbb{P}^{m} \Longrightarrow \exists \; \mathbb{P}^{n} \; . \ni . \; f \in \mathbb{P}^{n}$; and, $f \in \mathbb{P}^{n} \Longrightarrow Lf = Qf$; hence, $Lf = Qf$ on $\mathbb{P}^{\infty}$. Conversely, now *suppose* $f \in \mathbb{P}^{\infty} \Longrightarrow Lf = Qf$; then, $\forall \; \mathbb{P}^{m}$, $\mathbb{P}^{m} \subset \mathbb{P}^{\infty}$; hence, for *arbitrary* $\mathbb{P}^{n}$: $f \in \mathbb{P}^{n} \Longrightarrow f \in \mathbb{P}^{\infty} \Longrightarrow Lf = Qf$. □

Theorem 4.01. *(Crafting a p-Binomial Sequence* $\mathbb{B}_m$ *for its unique Operator L).* Given an *integer* $q > 0$ and a *n*[th]*-order, (udd) Linear Operator L* $= \sum_{k=1}^{n} \lambda_k D^k$, $(\lambda_0 \neq 0 \neq \lambda_n)$ then, there exists: *(i) an Operator* $Q = \sum_{j=1}^{m} \mu_j D^j$ $(\mu_0 \neq 0 \neq \mu_m)$, and *a subspace* $\mathbb{P}^{m}$ $(m > q)$ such that $f \in \mathbb{P}^{m+1} \Longrightarrow (LQ)f = Df$; and also, *(ii) a p-Binomial, Sequence* $\mathbb{B}_m = \{p_k\}_{k=0}^{m}$ such that: $p_0(x) = 1$ and $p_k(x) = \int_0^x Q[kp_k(t)]\,dt$, $1 \leq k \leq m$.

Proof. *Suppose the hypothesis.* Then, application of *Lemma L4.01* asserts there exist an *Operator* $Q = \sum_{j=1}^{m} \mu_j D^j$ $(\mu_0 \neq 0 \neq \mu_m)$, and *a subspace* $\mathbb{P}^{m}$ $(m > q)$ such that $f \in \mathbb{P}^{m+1} \Longrightarrow (LQ)f = Df$. Now, note that: *(a) L and Q commute,* $LQ = QL$; and

(b) $L = \sum_{k=1}^{n} \lambda_k D^k = \left(\sum_{k=1}^{n} \lambda_k D^{k-1} \right) D = L_1 D$. Hence, $f \in \mathbb{P}^{m+1} \Longrightarrow Df = (LQ)f$

$\Longrightarrow Df = (QL)f = (Q(L_1 D))f = (QL_1)[Df] \Longrightarrow I[Df] = (QL_1)[Df]$. Thus, $f \in \mathbb{P}^{m+1} \Longrightarrow [\,(LQ)f = Df = (QL)f$ <u>*and*</u> $(QL_1)[Df] = I[Df\,]$. Hence, (QL_1) is the *Identity Map* on $\mathbb{P}^{m}$, since $D[\mathbb{P}^{m+1}] = \mathbb{P}^{m}$. These notes and observations shall now be applied in order to create a *companion p-Binomial Sequence Basis* $\mathbb{B}_m = \{p_k\}_{k=0}^{m} \subset \mathbb{P}^{m}$ for the *Operator L*. This development proceeds a follows:

(4.5) $p_0(x) \equiv 1$, *by definition.* Then, *inductively define* $p_k(x)$, $1 \leq k \leq m$, by

$$Lp_k(x) = kp_{k-1}(x) \Longrightarrow (QL)p_k(x) = Q[\,kp_{k-1}(x)\,]$$

$$\Longrightarrow Dp_k(x) = Q[\,kp_{k-1}(x)\,] \Longrightarrow \boxed{p_k(x) = \int_0^x Q[\,kp_{k-1}(t)\,]\,dt} .$$

Hence, it is easily observed that this *basis* for $\mathbb{P}^m$ is indeed a *companion normal family basis* for the *Operator L* by applying the above *notes and observations* as follows: (1) $p_0 \in \mathbb{B}_m$, *where* $p_0(x) = 1$; (2) $p_k(0) = 0,\ 1 \le k \le m$; and (3) $Lp_k(x) = kp_{k-1}(x)$ as presented (below) here:

(4.6) $$Lp_k(x) = (L_1D)\left[\int_0^x Q[\,kp_{k-1}(t)\,]\,dt\right] = L_1[\,Q[\,kp_{k-1}(x)\,]\,] = (L_1Q)[\,kp_{k-1}(x)\,]$$

$$= kp_{k-1}(x)\,,$$ noting that $(L_1Q) = I =$ *Identity map* on $\mathbb{P}^m$.

Therefore, by *Theorem 2.02,* the *inductively defined basis* $\mathbb{B}_m = \{p_k\}_{k=0}^m \subset \mathbb{P}^m$ of Item (4.5) is the *unique companion normal family basis* for the *given Operator L* on the *subspace* $\mathbb{P}^m$. Consequently, the Item (4.5) *inductively crafted* the *companion normal family basis* $\mathbb{B}_m$ via a simple, direct constructive application of *property (iii)* found in *Theorem 2.02.* Moreover, given the *presentation form* of L, this *basis* $\mathbb{B}_m = \{p_k\}_{k=0}^m$ must be a *p-Binomial Sequence* by virtue of COROLLARY T3.03B. ☐

Observe that the *hypothesis* of the previous theorem (Theorem 4.01) is *sufficient* to actually *assert the existence* of an *unique, companion, p-Binomial Basis* for the *given Operator L* defined on $\mathbb{P}^\infty$; applications of *Theorem 2.04 and COROLLARY T3.03* clearly establish such assertion.

Noting the *unique* character of such basis, the *development of (4.5)* does *formulate construction devices (the Operator Q & the Fundamental Thm of calculus)* to render the *actual polynomials* which comprise that *unique companion p-Binomial Basis* for the *Operator L defined on subspace* $\mathbb{P}^m$.

Thus far, the collective developments of *Article 4* have presented the *foundation, procedures and formulations* which can be applied to construct the *actual polynomials* of the *companion, p-Binomial Sequence* paired to a *given (udd, polynomial differential) Linear Operator* $L = \sum_{k=1}^{\infty} \lambda_k D^k\ ,\ (\lambda_1 \ne 0)$. Next, the remaining discussions of this article shall be focused on establishing a foundation in order to devise an application that *can render* the *Linear Operator* $L = \sum_{k=1}^{n} \lambda_k D^k\ ,\ (\lambda_1 \ne 0)$, which is *paired to a given companion, p-Binomial Sequence.* Hence, in the spirit of *such advancement,* consider the theorem which next follows.

<u>Theorem 4.02</u>. *(Crafting an Operator L for its unique p-Binomial Sequence* $\mathbb{B}_n$*).* Given *a p-Binomial, Sequence* $\mathbb{B}_n = \{p_k\}_{k=0}^n$, then, there exists: a *unique n^th-order, companion (udd) Linear Operator* $L = \sum_{k=1}^{n} \lambda_k D^k$, $(\lambda_0 \ne 0 \ne \lambda_n)$ such that $Lp_k = kp_{k-1}\ ,\ 0 \le k \le n$.

<u>Proof</u>. Suppose the hypothesis. Then, via *COROLLARY T3.03,* an *unique such L exists* and *satisfies the recursive formula* for the elements of $\mathbb{B}_n$. Now, in order to determine the λ*-sequence values* for L, observe the properties of $\mathbb{B}_n$ as displayed by *Item (4.7).*

(4.7) $\mathbb{B}_n = \{p_k(x) \,\big|\, p_0(x) \equiv 1, \deg(p_k) = k$ and $p_k(0) = 0\,,\; 1 \le k \le n\}\,.$

Applications of the *recursion* $Lp_k(x) \,=\, kp_{k-1}(x)\,,\, 1 \le k \le n,$ and the $\mathbb{B}_n$ *properties* render:

(4.8a) $1 \,=\, Lp_1(x)\Big|_{x=0} \,=\, \lambda_1 p_1'(0)$

$0 \,=\, Lp_2(x)\Big|_{x=0} \,=\, \lambda_1 p_2'(0) \,+\, \lambda_2 p_2''(0)$

$0 \,=\, Lp_3(x)\Big|_{x=0} \,=\, \lambda_1 p_3'(0) \,+\, \lambda_2 p_3''(0) \,+\, \lambda_3 p_3^{(3)}(0)$

.

.

$0 \,=\, Lp_n(x)\Big|_{x=0} \,=\, \lambda_1 p_n'(0) \,+\, \lambda_2 p_n''(0) \,+\, \lambda_3 p_n^{(3)}(0) \,+\, \ldots \,+\, \lambda_n p_n^{(n)}(0)$

Observing that the *system of linear equations of (*$4.8a$*)* has the appearance

(4.8b) $\mathbf{e_1} = \Pi_0 \Lambda$; where $\mathbf{e_1} = \boldsymbol{col}[1, 0, 0, \ldots, 0]$; $\Pi_0 = [\,p_k^{(j)}(0)]$; *and,* $\Lambda = \boldsymbol{col}[\lambda_1, \lambda_2, .. \lambda_j, .. \lambda_n]$ *;*

and that: *(i)* the *(n × n)-coefficient matrix* Π_0 is *lower triangular* with *non-zero diagonal elements*, and that *(ii)* $\mathbf{e_1} \ne 0$, then there *exists a unique solution vector* Λ *for this system (which supports the unique character of L).* Appealing to the *lower triangular (row-echelon) form* of (4.8a), and then proceeding with *forward substitution* prescribed by the following *recursive relations* yields the *numerical values* of the Λ*-components.* Next, observe that,

(4.9a) $\lambda_1 \,=\, \dfrac{1}{p_1'(0)}$, and

(4.9b) $\lambda_k \,=\, \dfrac{(-1)}{p_k^{(k)}(0)} \displaystyle\sum_{j=1}^{k-1} \lambda_j p_k^{(j)}(0)\,,\; 2 \le k \le n\,.$

Thus, *Item devices (4.9a & 4.9b)* hereby render the Λ*-vector values* for the *unique, companion Operator L presentation* for a *given p-Binomial Sequence* $\mathbb{B}_n$. ☐

Article 5: p-Binomial Sequences of Order-m. In order to convey the notion of *higher order p-Binomial Sequences*, consider the following familiar model: the sequence of the *power function family,* $\mathbb{B} = \{p_k(x)\} = \{x^k\}$ is chosen to illustrate the very familiar *p-Binomial Sequence* of which the *companion (udd) Linear Operator is L = D (the ordinary derivative operator).* Now, imagine applying the *higher order* operator L^w to *each member of* $\mathbb{B}$*;* hence, create the *direct image* of $\mathbb{B}$ (with $k \ge w$ in order to avoid *zero vectors*) *via* L^w*;* thus consider

(5.1a) $L^w(\mathbb{B}) \,=\, \{L^w p_k \,\big|\, p_k \in \mathbb{B}\,,\, k \ge w\} = \{p_k^{(w)} \,\big|\, p_k \in \mathbb{B}\,,\, k \ge w\}\,.$

Therefore, the *illustrated (5.1a) model* $L^3(\mathbb{B})$ would render the following results.

(5.1b) $L^3(\mathbb{B}) = D^3(\{x^j\}) = \{p_j^{(3)} \mid p_j \in \mathbb{B}\,,\, j \geq 3\,\} = \{\frac{j!}{(j-3)!}x^{j-3} \mid j \geq 3\,\}$

$$= \left\{ \tfrac{3!}{0!}x^0\,,\, \tfrac{4!}{1!}x^1\,,\, \tfrac{5!}{2!}x^2\,,\, \tfrac{6!}{3!}x^3\,, \ldots,\, \tfrac{j!}{(j-3)!}x^{j-3}\,,\, \ldots \right\}\ .$$

Higher order p-Binomial Sequences (as shall next be explored) actually satisfy an *extended p-Binomial Expansion-Identity (of Order-w)* which relates *initial, finite subsequences* of $L^w(\mathbb{B})$ by satisfying a similar *sum expansion-identity profile* that closely resembles the *p-Binomial Expansion-Identity* presented in the preceding articles. The above notation appearing in *Item (5.1)* shall be adopted to reference *repeated operator applications* applied *elements* $p_k \in \mathbb{B}$. Hence, observe the following *relational details* appearing in *Item (5.2), below*.

(5.2) *(a)* $Lp_k = kp_{k-1}$; *(b)* $L^w p_n = \frac{n!}{(n-w)!}\, p_{n-w} = p_n^{(w)}$;

(c) $(n-k)!\, p_{n-w-k} = (n-w-k)!\, p_{n-k}^{(w)}$; *(d)* $(k+w)!\, p_k = k!\, p_{k+w}^{(w)}$.

Moreover, unless otherwise stated, the *symbol* $\mathbb{B}$ shall designate the *p-Binomial Sequence* for the *(udd, Shift-Invariant) Linear Operator* $L = \sum\limits_{k=1}^{\infty} \lambda_k D^k$, $(\lambda_1 \neq 0)$, and such that the *two of which* constitute a *(unique, each to the other) companion pair* $\langle L\ , \mathbb{B}\rangle$. Now, regarding the preceding, introductory pronouncements: *given a companion pair* $\langle L\ , \mathbb{B}\rangle$,
as above described, let $w \geq 0$ be a *given integer;* then: $a \in \mathbb{R}$ *(real numbers)* $\Longrightarrow$

(5.3a) $p_n^{(w)}(a+x) = L^w p_n(a+x) = \frac{n!}{(n-w)!}\, p_{n-w}(a+x)$

$$= \frac{n!}{(n-w)!} \sum_{k=0}^{n-w} \binom{n-w}{k} p_{n-w-k}(a)\, p_k(x) = \sum_{k=0}^{n-w} \frac{n!}{(n-w)!} \binom{n-w}{k} p_{n-w-k}(a)\, p_k(x)$$

$$= \sum_{k=0}^{n-w} \frac{n!}{(n-w)!} \binom{n-w}{k} p_{n-w-k}(a)\, p_k(x)$$

$$= \sum_{k=0}^{n-w} \frac{n!}{(n-w)!}\ \frac{(n-w)!}{k!(n-w-k)!} \left[\frac{(n-w-k)!}{(n-k)!}\, p_{n-k}^{(w)}(a) \right] \left[\frac{k!}{(k+w)!}\, p_{k+w}^{(w)}(x) \right]$$

$$= \sum_{k=0}^{n-w} \frac{k!}{(k+w)!} \binom{n}{k} p_{n-k}^{(w)}(a)\ p_{k+w}^{(w)}(x)\ ;\ \textit{and, consequently}$$

(5.3b) $$\boxed{p_n^{(w)}(a+x) = \sum_{k=0}^{n-w} \frac{k!}{(k+w)!} \binom{n}{k} p_{n-k}^{(w)}(a)\ p_{k+w}^{(w)}(x)}\ ;$$

$(\ p_n^{(w)} \in L^w(\mathbb{B})\ , n \geq w)$.

The developmental details of *Item (5.3a)* engaged the *companion properties* of the *pair* $\langle L\ , \mathbb{B}\rangle$, and *substitutions* from the *relational details (5.2c &d).* The *extended expansion identity (5.3b)* could be referenced as the *p-Binomial Expansion Identity of Order-w.* Clearly, this *Identity (5.3b)* of *order zero* is simply the *p-Binomial Identity* discussed in the preceding articles here in this manuscript.

A brief digression to *Article 4* reveals the *formulation* of *construction devices* which can be implemented to *craft* the *actual paired components* of *system models* $\langle L\,,\mathbb{B}\rangle$. In particular, *Theorem 4.01* provided the foundational structure to formulate *Item (4.5)* to craft the *model elements of* $\mathbb{B}$. In similar fashion, such *model crafting* can also be accomplished regarding the construction of $L^w(\mathbb{B})$. Hence, *suppose the hypothesis* of *Theorem 4.01;* now, apply the theorem so that the *consequent Items (i) and (ii)* are satisfied. Then, observe the development presented in *Item (5.4):*

(5.4) $p_w^{(w)}(x) \equiv 1$; then, *inductively define* $p_k^{(w)}$, $w < k \leq q < m$, $p_k^{(w)} \in L^w(\mathbb{B}_m)$, as below detailed.

$$Lp_k(x) = kp_{k-1}(x) \implies (QL)p_k(x) = Q[\,k\,p_{k-1}(x)\,]$$

$$\implies L^w Dp_k(x) = L^w Q[\,k\,p_{k-1}(x)\,] \implies Dp_k^{(w)}(x) = Q[\,k\,p_{k-1}^{(w)}(x)\,]$$

$$\implies \boxed{p_k^{(w)}(x) = \int_0^x Q[\,k\,p_{k-1}^{(w)}(t)\,]\,dt} \ . \quad \square$$

Appeal to *Items (4.5) & (5.4)* renders the *rather novel Item (5.5):*

(5.5) $$\frac{\partial^w}{\partial x^w}\int_0^x Q[\,k\,p_{k-1}(t)\,]\,dt = p_k^{(w)}(x) = \int_0^x Q[\,\tfrac{\partial^w}{\partial t^w}\,k\,p_{k-1}(t)\,]\,dt \ .$$

<u>Article 6: p-Binomial Sequence Illustrative Modeling Exercises</u>. This article offers *attached reference pages* which offer *actual implementations* that *illustrate* how to devise and construct *p-Binomial Sequences* $\mathbb{B}$ and/or *Linear Operators L* (the two of) which *constitute a unique, (each to the other) companion pair* $\langle L\,,\mathbb{B}\rangle$.

<u>MODEL #01</u>. Suppose the desire is to *construct* a *p-Binomial Sequence Model.*
First, apply *Theorem 4.01* of *Article 4.* Hence, the following *steps are reviewed:*

(a)→ *create* $L = \sum_{k=1}^{n} \lambda_k D^k$ $(\lambda_1 \neq 0 \neq \lambda_n)$ so that $Q = \sum_{j=0}^{m} \mu_j D^j$ $(\mu_0 \neq 0 \neq \mu_m)$ can be constructed via the *Division Algorithm.* However, examination of the *proof of <u>LEMMA L4.01</u>* reveals that L and Q form a *unique pair;* so, the <u>*Division Algorithm* will *create each* from the *other*</u>*!*

(b)→ next, from *(01a)*, *simply create Q* by <u>*declaration*</u>*;* then, *if* L is needed *and/or* desired, ... then use the <u>*Division Algorithm*</u> to construct it!

(c)→ so, ... *Declare Q* = $e^D = \sum_{j=0}^{\infty} \frac{1}{j!} D^j = (1 + D + \frac{1}{2!}D^2 + \frac{1}{3!}D^3 + \ldots)$.

(d)→ now, *if* L *is desired*, ... then, use the *DA* to *develop it:* $LQ = D$,

hence: $L = \frac{D}{Q} = \frac{D}{e^D} = De^{-D} = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} D^{j+1}$ *;* thus,

$L = (D - D^2 + \frac{1}{2!}D^3 - \frac{1}{3!}D^4 + \ldots)$.

(e)→ Now, ... *verify that:* $LQ = D$. So, ... $LQ = (De^{-D})(e^D) = D$.
Hence, ... *yes, it checks!!!*

(f)→ Please *Proceed to* MODEL #01 (found on the following pages).

(Note: This model *rendered* the *Abel Series:* $p_n(x) = x(x - an)^{n-1}$, *with* $a = -1$*;* and, the *Operator* $L = De^{-D}$).

MODEL #02. Suppose the desire is to *construct the unique, companion Linear Operator L (Model)* for a *given p-Binomial Sequence*. Then, *Theorem 4.02* of *Article 4* is implemented. Hence, the following *steps are reviewed:*

(a)→ the *companion properties* of the $\mathbb{B}_n$ and L are applied; hence, note that

(b)→ $p_0(x) = 1$; deg(p_k) $= k$; and, $p_k(0) = 0$, $1 \leq k \leq n$.

(c)→ $Lp_k = kp_{k-1}$, $1 \leq k \leq n$.

(d)→ Next, *implement Items (4.9a & b)* in order to solve the *system:* $\Pi_0 \Lambda = \boldsymbol{e_1}$.

(e)→ Hence: *(4.9a)* $\lambda_1 = \frac{1}{p_1^{(1)}(0)}$; and,

(4.9b) $\lambda_k = \frac{(-1)}{p_k^{(k)}(0)} \sum_{j=1}^{k-1} \lambda_j p_k^{(j)}(0)$, $2 \leq k \leq n$.

(f)→ Formulate: $L = \sum_{k=1}^{n} \lambda_k D^k$, $(\lambda_0 \neq 0 \neq \lambda_n)$.

(Note: This model was *illustrated using* the decreasing factorial powers defined by *the product of the n-decreasing factors:* $x^{(n)} = x(x-1)(x-2)\ldots(x-n+1)$ *;* this model used the *finite p-Binomial Sequence* $\mathbb{B}_7$ (of degree ≤ 7); for this *p-Binomial Sequence,* the *Operator L* = $e^D - 1 = \Delta =$ *Newton's (unit) advancing difference Operator*).

MODEL #03. Suppose the desire is to *create* a *companion pair* $\langle L , \mathbb{B} \rangle$*-Model* in order to further *create and illustrate* the *companion properties interplay between the Order-m companion pair* L *and* $L^w(\mathbb{B})$. Hence, the following *steps are reviewed:*

(a)→ First, *implement the methods of Model #01* to *create the* $\langle L , \mathbb{B} \rangle$*-Model.*

(b)→ Next, create the *Operator Q ;* then, devise $\{ p_k \}$ *;* then, *apply (5.5)* in order to craft $p_k^{(w)}(x)$ as in *(c), below:*

(c)→ $p_k^{(w)}(x) = \frac{\partial^w}{\partial x^w} \int_0^x Q[\, k\, p_{k-1}(t) \,]\, dt$.

(d)→ Next, confirm that $Lp_k^{(w)}(x) = kp_{k-1}^{(w)}(x)$ for the *Model*; then,

(e)→ *Verify* the *p-Binomial Expansion Identity of Order-w* for the *Model, Order-w Expansion Identity (5.3b):*

$$\boxed{p_n^{(w)}(a+x) \;=\; \sum_{k=0}^{n-w} \frac{k!}{(k+w)!}\binom{n}{k}\, p_{n-k}^{(w)}(a)\; p_{k+w}^{(w)}(x)} \;;\qquad (\, p_n^{(w)} \in L^w(\mathbb{B})\,,\, n \geq w).$$

(Note: This model was *illustrated by using* the <u>increasing factorial powers</u> defined by *the product of the n-increasing factors:* $x^{[n]} = x(x{+}1)(x{+}2)...(x{+}n{-}1)$ *;* this model used the *finite p-Binomial Sequence* $\mathbb{B}_7$ (of degree ≤ 7); for this *p-Binomial Sequence,* the *Operator L* = $1 - e^{-D} = \nabla\; =$ *Newton's (unit) backward difference Operator*).

<u>MODEL #04</u>. This *Model #04* explores and illustrates *novel creations* and *explorations* regarding imaginative *designs* such as: (a)--*Operators L and/or Q* where *Fibonacci sequence-values* appear as the λ*-coefficients*, or the μ*-coefficients;* (b)--inspect such *companion* p_k*-Binomial Sequences;* (c)--explore the *p-Binomial Sum-Expansion Identities.* Illustrations of such *model entities* are displayed via *Tables T1, T2 and T3* as presented.

<u>MODEL #05</u>. Further *novel applications* of preceding structural developments include: (a)--Declaring *Operator L = sin*(D) *= (Maclaurin series in D);* (b)--then, *formulate Q* $= \frac{D}{sin(D)}$; (c)--*verify the p-Binomial Expansion-Sum* for the *companion* p_k*-Binomial Sequence;* (d)--clearly, other *creative crafting* of *Operators L and Q* are abundant.

<u>MODEL #06</u>. One final *novel model* of preceding structural developments is devised by implementing the δ*-sequence* $\{\delta_k\}$ *defined* by the *recursion:* $\delta_k \;=\; \frac{1}{2^{2k}}\binom{2k}{k}$; interestingly, $(1{-}x)^{-\frac{1}{2}} = \sum_{k=0}^{\infty}\delta_k x^k$; and, $\sum_{k=0}^{n}\delta_k \;=\; 2n\delta_n.$ This δ*-sequence* $\{\delta_k\}$ has *many other such interesting properties.* Hence, *incorporating* this δ*-sequence* into the composition of *Operators L and/or Q* is somewhat intriguing. In particular devise: (a)--*Operators L and/or Q* where δ*-sequence-values* appear as the λ*-coefficients*, or the μ*-coefficients;* (b)--inspect such *companion,* p_k*- and* q_k*-* , p*-Binomial Sequences;* and, then (c)--explore the *companion p-Binomial Sum-Expansion Identities.* Illustrations of such *model entities* are displayed via *Tables T1, T2 and T3* as presented.

Illustrative Models
of
p-Binomial Sequences

Models Pages -->

Article 6. MODEL #01.

Objective: *Declare Q; Devise L; Create* p_k *–sequence* B_M *;*
Illustrate $Lp_k = k\,p_{k-1}$; Model p-Binomial Sum-Expansion.

```
Clear[d, OpL, OpQ , L , Q , p, B, L]

OpQ = Series[E^d, {d, 0, 12} ] (*  Operator Q ; Q = e^D (declared)  *)
OpL = Series[d/OpQ , {d, 0, 12 } ]  (*  Operator L ; LQ = D ==> L = D/Q *)
```

$$1 + d + \frac{d^2}{2} + \frac{d^3}{6} + \frac{d^4}{24} + \frac{d^5}{120} + \frac{d^6}{720} + \frac{d^7}{5040} + \frac{d^8}{40\,320} + \frac{d^9}{362\,880} + \frac{d^{10}}{3\,628\,800} + \frac{d^{11}}{39\,916\,800} + \frac{d^{12}}{479\,001\,600} + O[d]^{13}$$

$$d - d^2 + \frac{d^3}{2} - \frac{d^4}{6} + \frac{d^5}{24} - \frac{d^6}{120} + \frac{d^7}{720} - \frac{d^8}{5040} + \frac{d^9}{40\,320} - \frac{d^{10}}{362\,880} + \frac{d^{11}}{3\,628\,800} - \frac{d^{12}}{39\,916\,800} + O[d]^{13}$$

```
(OpQ) (OpL) (* VERIFY that LQ = D + R(D) , where minimal-deg[R]≥13  *)
```

```
d + O[d]^13 (* This product-OUTPUT verifies minimal-deg[R]≥13   *)
```

$$Q[f\_] := f + \sum_{j=1}^{12} \frac{1}{j\,!}\,\partial_{\{t,j\}} f \quad (*\ \ \text{Operator Q}\ *)$$

$$L[f\_] := \sum_{k=1}^{12} \frac{(-1)^{k-1}}{(k-1)\,!}\,\partial_{\{x,k\}} f \quad (*\ \ \text{Operator L}\ *)$$

```
M = 9; (*  M-value DELCARED for a finite p-Binomial Sequence  B_M *)
p[0, x_] := 1 ;  (*  p_0(x)=1   *)
```

$$p[k\_, x\_] := \int_0^x Q[k * p[k-1, t]]\, dt \quad (*\ \ \text{Recursion for } p_k(x)\ \ *)$$

In[▫]:=

```
Print[ "p_k(x)" , "        " , "Lp_k(x)= kp_{k-1}(x)"]
T1 = Table[ Factor[{p[k, x] , L[ p[k, x] ] }], {k, 0, M} ];
Print[ TableForm [T1] ] (* Sequence Tables Confirm: Lp_k(x)= k p_{k-1}(x)
    -- TABLE T1 ----------    *)
```

$p_k(x)$	$Lp_k(x) = kp_{k-1}(x)$
1	0
x	1
x (2 + x)	2 x
$x\,(3+x)^2$	3 x (2 + x)
$x\,(4+x)^3$	$4\,x\,(3+x)^2$
$x\,(5+x)^4$	$5\,x\,(4+x)^3$
$x\,(6+x)^5$	$6\,x\,(5+x)^4$
$x\,(7+x)^6$	$7\,x\,(6+x)^5$
$x\,(8+x)^7$	$8\,x\,(7+x)^6$
$x\,(9+x)^8$	$9\,x\,(8+x)^7$

In[ ]:=

```
BiExSum[N_] := Sum[Binomial[N, k] p[N - k, a] p[k, x], {k, 0, N}] (* p-Binomial Expansion Sum *)
Print[ "pk(a+x)" , "        =        " , "BiExSum[k]"]
T2 = Table[ Factor[ { p[k, a + x] , BiExSum[k] } ], {k, 0, M} ];
(* pk-sequence = { pk(x) } *)
Print[ TableForm [T2] ] (* Sequence Tables Confirm: pn(a+x)= BiExSum[n]
   -- TABLE T2 ----------    *)
```

$p_k(a+x)$ = BiExSum[k]

1	1
$a + x$	$a + x$
$(a + x)(2 + a + x)$	$(a + x)(2 + a + x)$
$(a + x)(3 + a + x)^2$	$(a + x)(3 + a + x)^2$
$(a + x)(4 + a + x)^3$	$(a + x)(4 + a + x)^3$
$(a + x)(5 + a + x)^4$	$(a + x)(5 + a + x)^4$
$(a + x)(6 + a + x)^5$	$(a + x)(6 + a + x)^5$
$(a + x)(7 + a + x)^6$	$(a + x)(7 + a + x)^6$
$(a + x)(8 + a + x)^7$	$(a + x)(8 + a + x)^7$
$(a + x)(9 + a + x)^8$	$(a + x)(9 + a + x)^8$

Article 6. MODEL #02.

Objectives: Implement (4.9a & b) to determine **Λ**-*values* ; Formulate L from the **Λ**-*values* ; Confirm that $Lp_k = k\, p_{k-1}$; Verify the p_k-*Binomial Sequence* satisfies the *p-Binomial-Sum Identity.*

```
p[0, x_] := 1 ; (*  TABLE-T1 is the given p-Binomial Sequence B7  *)
p[n_, x_] := p[x, n] = (x - n + 1) p[n - 1, x]
T1 = Table[ p[k, x] , {k, 0, 7} ];
Print[ TableForm[ T1] ]  (*
---------------- TABLE-T1 -----------------   *)
```

```
1
x
(-1 + x) x
(-2 + x) (-1 + x) x
(-3 + x) (-2 + x) (-1 + x) x
(-4 + x) (-3 + x) (-2 + x) (-1 + x) x
(-5 + x) (-4 + x) (-3 + x) (-2 + x) (-1 + x) x
(-6 + x) (-5 + x) (-4 + x) (-3 + x) (-2 + x) (-1 + x) x
```

```
c[1] = 1;
c[n_] := ((-1 / ∂{x,n} p[n, x]) * Sum[c[k] * ∂{x,k} p[n, x], {k, 1, n-1}]) /.
  x -> 0 (* Determine Λ-coefficients for D *)
```

$\{1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \frac{1}{720}, \frac{1}{5040}\}$ (* Λ-coefficients *)

```
L[f_] := Sum[c[k] * ∂{x,k} f, {k, 1, 7}]  (*  Define Operator = L using the Λ-coefficients *)
Print[ "pk (x)" , "                                         " , "Lpk (x)  = k pk (x)"]
T2 = Table[ { p[k, x] , Factor[ L[ p[k, x]] ] }, {k, 0, 7}];
Print[ TableForm [ T2] ]  (*
__________________________TABLE-T2___________________________________  *)
```

$p_k(x)$	$Lp_k(x) = k\ p_k(x)$
1	0
x	1
(-1 + x) x	2 x
(-2 + x) (-1 + x) x	3 (-1 + x) x
(-3 + x) (-2 + x) (-1 + x) x	4 (-2 + x) (-1 + x) x
(-4 + x) (-3 + x) (-2 + x) (-1 + x) x	5 (-3 + x) (-2 + x) (-1 + x) x
(-5 + x) (-4 + x) (-3 + x) (-2 + x) (-1 + x) x	6 (-4 + x) (-3 + x) (-2 + x) (-1 + x) x
(-6 + x) (-5 + x) (-4 + x) (-3 + x) (-2 + x) (-1 + x) x	7 (-5 + x) (-4 + x) (-3 + x) (-2 + x) (-1 + x) x

```
BiExSum[n_] := Sum[Binomial[n, k] p[n - k, a] p[k, x], {k, 0, n}]
Print[ "pn(a+x)" , "                         =                    " , "BiExSum[n]"]
T3 = Table[ { p[n, a + x] , Factor[ BiExSum[n] ] }, {n, 0, 5}];
Print[ TableForm [ T3] ] (*
_____________________________ TABLE-T3 _____________________________________  *)
```

$p_n(a+x)$ = BiExSum[n]

1	1
a + x	a + x
(−1 + a + x) (a + x)	(−1 + a + x) (a + x)
(−2 + a + x) (−1 + a + x) (a + x)	(−2 + a + x) (−1 + a + x) (a + x)
(−3 + a + x) (−2 + a + x) (−1 + a + x) (a + x)	(−3 + a + x) (−2 + a + x) (−1 + a + x) (a + x)
(−4 + a + x) (−3 + a + x) (−2 + a + x) (−1 + a + x) (a + x)	(−4 + a + x) (−3 + a + x) (−2 + a + x) (−1 + a + x) (a + x)

Article 6. MODEL #03.

p-Binomial Expansion Identity of Order-w

$$L^w[p_n(x+y)] = \sum_{k=0}^{n-w} \left(\frac{k!}{(k+w)!}\right) \binom{n}{k} L^w[p_{n-k}(x)]\, L^w[p_{k+w}(y)].$$

Note that L^0 denotes the *Identity Operator.* Here, *L* denotes the *shift-invariant Operator* for the *p-Binomial Sequence* $\{p_k\}$.

==

Objectives: Create Operators *L & Q; develop* $\{p_k\}$; Devise $p_k^{(w)}$; Verify that $Lp_k^{(w)} = k\,p_{k-1}^{(w)}$; Verify the above *Order–w Identity.*

```
L = Series[1 - E^-d , {d, 0, 9}]  (*  DECLARED Operator L  *)
Q = Series[ d/(1 - E^-d) , {d, 0, 9} ]  (*  Operator Q formulated: d = L Q + R *)
```

$$d - \frac{d^2}{2} + \frac{d^3}{6} - \frac{d^4}{24} + \frac{d^5}{120} - \frac{d^6}{720} + \frac{d^7}{5040} - \frac{d^8}{40\,320} + \frac{d^9}{362\,880} + O[d]^{10}$$

$$1 + \frac{d}{2} + \frac{d^2}{12} - \frac{d^4}{720} + \frac{d^6}{30\,240} - \frac{d^8}{1\,209\,600} + O[d]^{10}$$

```
Clear[L, Q, LL, Lwpn, B, p]
L[f_] := Sum[(-1)^(k-1)/k! ∂{x,k} f, {k,1,7}]  (*  Operator L program-code  *)
Q[f_] := f + 1/2 ∂{t,1} f + 1/12 ∂{t,2} f - 1/720 ∂{t,4} f + 1/30240 ∂{t,6} f - 1/1209600 ∂{t,8} f
(*  Operator Q program code  *)
p[0, x_] := 1
p[n_, x_] := ∫_0^x Q[ n * p[n - 1, t ]] ⅆt (* p-Binomial Sequence  *)
Print["---------------- TABLE-T1 -------------------"]
Print[ "pk (x)" , "                                        " , "Lpk (x)"]
 T1 =  Table[  Factor[{p[k, x] , L[ p[k, x] ] } ], {k, 0, 7} ]  ;
Print[ TableForm[T1] ]
```

---------------- TABLE-T1 --------------------

$p_k(x)$	$Lp_k(x)$
1	0
x	1
x (1 + x)	2 x
x (1 + x) (2 + x)	3 x (1 + x)
x (1 + x) (2 + x) (3 + x)	4 x (1 + x) (2 + x)
x (1 + x) (2 + x) (3 + x) (4 + x)	5 x (1 + x) (2 + x) (3 + x)
x (1 + x) (2 + x) (3 + x) (4 + x) (5 + x)	6 x (1 + x) (2 + x) (3 + x) (4 + x)
x (1 + x) (2 + x) (3 + x) (4 + x) (5 + x) (6 + x)	7 x (1 + x) (2 + x) (3 + x) (4 + x) (5 + x)

```
LL[L_, f_, w_] := Nest[L, f, w] (*  Operator L^w program-code  *)
Lwpk[w_, k_, z_] := Factor[ LL [L, p[k, x], w] ] /. x → z
(*  polynomial L^w p_k program-code *)

B[w_, n_] := Sum[ (j! / (j + w)!) Binomial[n, j] Lwpk[w, n - j, x] Lwpk[w, j + w, y], {j, 0, n-w} ]
(*  p-Binomial Identity Order-w  program-code  *)

n = 7;  w = 2; (*  B_n=B_7={p_1, p_2,..., p_7}  & Order-w = 2  *)
Print[ "p_k^(2)(x)" , "                  " , "L[p_k^(2)(x)] = k p_k-1^(2)(x)"]
T2 = Table[  Factor[{ Lwpk[w, k, x] , L[ Lwpk[w, k, x] ] } ], {k, 1, n} ]  ;
Print[ TableForm [T2] ] (* Verification below: (*
  ---------------- TABLE-T2 -----------------   *)
```

$p_k^{(2)}(x)$	$Lp_k^{(2)}(x) = k\, p_{k-1}^{(2)}(x)$
0	0
2	0
6 x	6
12 x (1 + x)	24 x
20 x (1 + x) (2 + x)	60 x (1 + x)
30 x (1 + x) (2 + x) (3 + x)	120 x (1 + x) (2 + x)
42 x (1 + x) (2 + x) (3 + x) (4 + x)	210 x (1 + x) (2 + x) (3 + x)

```
(*  Below, p_n^(2)(x+y)  = B[w,n]  = Order-w=2 Expansion Identity  *)
Print[ "p_n^(2)(x+y)" , "                          " , "B[2,n]"]
T3  = Table[ Factor[{ Lwpk[w, n, x + y] , B[2, n] } ], {n, 1, 6} ] ;
Print[ TableForm [ T3] ]  (* Verification that: p_n^(2)(x+y)  = B[2,n])
   -------------- TABLE-T3 --------------   *)
```

$p_n^{(2)}(x+y)$	B[2,n]
0	0
2	2
6 (x + y)	6 (x + y)
12 (x + y) (1 + x + y)	12 (x + y) (1 + x + y)
20 (x + y) (1 + x + y) (2 + x + y)	20 (x + y) (1 + x + y) (2 + x + y)
30 (x + y) (1 + x + y) (2 + x + y) (3 + x + y)	30 (x + y) (1 + x + y) (2 + x + y) (3 + x + y)

(* Verification that: $p_n^{(2)}(x+y)$ = B[2,n] *)

Article 6. MODEL #04.

Features of this Model #04. (a)--Illustrations of *Novel Applications* of the structural developments appearing in the preceding *Articles;* hence, *select L = [Fibonacci sequence- values for the λ-coefficients];* (b)--determine the *Q_operator* and craft the p_n sequence ; (c)--Verify that $Lp_k = k\, p_{k-1}$ for these *Model normal family basis elements* as displayed in *Table-T1 ;* (d)--next, *Create an Operator_Q = [Fibonacci sequence-values for the μ-coefficients];* then, create the *companion_ L so that: LQ = D ;* (e)--apply the newly created *Q and L* and fashion the q_k *–Binomial Sequence* displayed in *Table-T2;* (f)--next, *observe Table-T3* which *VERIFIES* that q_k *satisfies the p-Binomial Expansion Sum Identity.*

```
Clear[L, Q, p , f]
c[1] = 1; c[2] = 1;
c[n_] := c[n] = c[n - 1] + c[n - 2] (*  Create Fibonacci Sequence terms  *)
Table[c[n] , {n, 1, 13}]; (*  First 13-terms of the Fibonacci Sequence  *)
L_oper = Series[Sum[c[k] d^k, {k, 1, 13}] , {d, 0, 13} ]
Q_oper = Series[d/(Sum[c[k] d^k, {k, 1, 13}]) , {d, 0, 13} ]
```

$d + d^2 + 2 d^3 + 3 d^4 + 5 d^5 + 8 d^6 + 13 d^7 + 21 d^8 + 34 d^9 + 55 d^{10} + 89 d^{11} + 144 d^{12} + 233 d^{13} + O[d]^{14}$

(* L-Fibonacci Series Profile *)

$1 - d - d^2 + 377 d^{13} + O[d]^{14}$ (* Q_operator companion profile *)

```
L[f_] := Sum[c[k] ∂{x,k} f, {k, 1, 13}] (*  L_Operator code  *)
Q[f_] := f - ∂{t,1} f - ∂{t,2} f + 377 ∂{t,13} f (*  Q_Operator code  *)
p[0, x_] := 1
p[k_, x_] := Integrate[Q[ k p[k - 1, t] ], {t, 0, x}]
```

```
Print[ "pk(x)" , "                                        " , "Lpk(x)"]
T1 = Table[ Factor[{ p[k, x] , L[ p[k, x]] } ], {k, 0, 6}];
Print[ TableForm[ T1] ]  (*
_________________________ TABLE-T1 ______________________ *)
```

$p_k(x)$	$Lp_k(x)$
1	0
x	1
$(-2+x)\,x$	$2x$
$(-6+x)\,x^2$	$3\,(-2+x)\,x$
$x\,(48+12x-12x^2+x^3)$	$4\,(-6+x)\,x^2$
$x\,(-360+240x+60x^2-20x^3+x^4)$	$5x\,(48+12x-12x^2+x^3)$
$x\,(-720-3600x+600x^2+180x^3-30x^4+x^5)$	$6x\,(-360+240x+60x^2-20x^3+x^4)$

```
Clear[Q, L, q]
Q[f_] := Sum[c[k] ∂{t,k-1} f, {k, 1, 13}]   (* Q-Fibonacci Operator code  *)
L[f_] := ∂{x,1} f - ∂{x,2} f - ∂{x,3} f   (*  companion L_Operator code  *)
q[0, x_] := 1
q[k_, x_] := Integrate[Q[ k q[k - 1, t] ], {t, 0, x}]
```

```
Print[ "qk(x)" , "                                                  " , "Lqk(x)"]
T2 = Table[ Factor[{ q[k, x] , L[ q[k, x]] } ], {k, 0, 6}];
Print[ TableForm [ T2] ] (*
_________________________TABLE-T2______________________ *)
```

$q_k(x)$	$Lq_k(x)$
1	0
x	1
$x\,(2+x)$	$2\,x$
$x\,(18+6\,x+x^2)$	$3\,x\,(2+x)$
$x\,(240+84\,x+12\,x^2+x^3)$	$4\,x\,(18+6\,x+x^2)$
$x\,(4560+1560\,x+240\,x^2+20\,x^3+x^4)$	$5\,x\,(240+84\,x+12\,x^2+x^3)$
$x\,(110\,880+37\,800\,x+5880\,x^2+540\,x^3+30\,x^4+x^5)$	$6\,x\,(4560+1560\,x+240\,x^2+20\,x^3+x^4)$

```
BiExSum[N_] := Sum[Binomial[N, k] q[N - k, a] q[k, x], {k, 0, N}] (* p-Binomial Expansion Sum *)
Print[ "qk(a+x)" , "                                                  " , "BiExSum[k]"]
T3 = Table[ Factor[ { q[k, a + x] , BiExSum[k] } ], {k, 0, 4} ]; (* qk(a+x) = BiExSum[k] *)
Print[ TableForm [T3] ] (*
____________________________ TABLE-T3 _________________________ *)
```

$q_k(a+x)$	BiExSum[k]
1	1
$a+x$	$a+x$
$(a+x)\,(2+a+x)$	$(a+x)\,(2+a+x)$
$(a+x)\,(18+6\,a+a^2+6\,x+2\,a\,x+x^2)$	$(a+x)\,(18+6\,a+a^2+6\,x+2\,a\,x+x^2)$
$(a+x)\,(240+84\,a+12\,a^2+a^3+84\,x+24\,a\,x+3\,a^2\,x+12\,x^2+3\,a\,x^2+x^3)$	$(a+x)\,(240+84\,a+12\,a^2+a^3+84\,x+24\,a\,x+3\,a^2\,x+12\,x^2+3\,a\,x^2+x^3)$

(Note that *Table-T3 does VERIFY* that the above q_k – sequence does satisfy the p-Binomial Expansion Sum)

Features of this Model #05. (a)--Illustrations of *Novel Applications* of the structural developments appearing in the preceding *Articles;* hence, *select L = Sin[D] (series expansion) as a p-Binomial Operator;*(b)--then, *examine the companion p-Binomial Sequence;* (c)--*Verify* that the *p-Binomial Expansion Identity is satisfied for Model entries.*

```
Clear[L, Q, p, T1, T2]
L_oper = Series[Sin[d] , {d, 0, 12}]
Q_oper = Series[d/Sin[d] , {d, 0, 12} ]
```

$$d - \frac{d^3}{6} + \frac{d^5}{120} - \frac{d^7}{5040} + \frac{d^9}{362880} - \frac{d^{11}}{39916800} + O[d]^{13}$$

$$1 + \frac{d^2}{6} + \frac{7 d^4}{360} + \frac{31 d^6}{15120} + \frac{127 d^8}{604800} + \frac{73 d^{10}}{3421440} + \frac{1414477 d^{12}}{653837184000} + O[d]^{13}$$

$$d - \frac{d^3}{6} + \frac{d^5}{120} - \frac{d^7}{5040} + \frac{d^9}{362880} - \frac{d^{11}}{39916800} + O[d]^{13}$$ (* L = [Sine-Series Operator] *)

$$1 + \frac{d^2}{6} + \frac{7 d^4}{360} + \frac{31 d^6}{15120} + \frac{127 d^8}{604800} + \frac{73 d^{10}}{3421440} + \frac{1414477 d^{12}}{653837184000} + O[d]^{13}$$ (* Q Companion *)

$$L[f\_] := \sum_{k=1}^{6} \frac{(-1)^{k-1}}{(2k-1)!} \partial_{\{x,2k-1\}} f$$ (* L Operator code *)

$$Q[f\_] := f + \frac{1}{6}\partial_{\{t,2\}} f + \frac{7}{360}\partial_{\{t,4\}} f + \frac{31}{15120}\partial_{\{t,6\}} f + \frac{127}{604800}\partial_{\{t,8\}} f + \frac{73}{3421440}\partial_{\{t,10\}} f$$

(* Q Operator code*)

p[0, x_] := 1; (* $p_0(x)=1$ *)

$$p[k\_, x\_] := \int_0^x Q[k*p[k-1, t]]\, dt$$ (* Recursion for $p_k(x)$ *)

In[68]:=
```
Print["----------------- TABLE-T1 -------------------"]
Print["pk(x)", "                                   ", "Lpk(x)"]
T1 = Table[ Factor[{p[k, x] , L[ p[k, x] ] }], {k, 0, 9} ];
Print[ TableForm [T1] ]  (* Sequences Table Confirms: Lpk(x) = pk(x) *)
```

----------------- TABLE-T1 -------------------

$p_k(x)$	$Lp_k(x)$
1	0
x	1
x^2	$2x$
$x(1+x^2)$	$3x^2$
$x^2(4+x^2)$	$4x(1+x^2)$
$x(1+x^2)(9+x^2)$	$5x^2(4+x^2)$
$x^2(4+x^2)(16+x^2)$	$6x(1+x^2)(9+x^2)$
$x(1+x^2)(9+x^2)(25+x^2)$	$7x^2(4+x^2)(16+x^2)$
$x^2(4+x^2)(16+x^2)(36+x^2)$	$8x(1+x^2)(9+x^2)(25+x^2)$
$x(1+x^2)(9+x^2)(25+x^2)(49+x^2)$	$9x^2(4+x^2)(16+x^2)(36+x^2)$

```
BiExSum[N_] := Sum[Binomial[N, k] p[N - k, a] p[k, x], {k, 0, N}] (* p-Binomial Expansion Sum *)
Print[ "---------------------------- TABLE-T2 ------------------------------"]
Print[ "p_k(a+x)" , "                    =                    " , "BiExSum[k]"]
T2 = Table[ Factor[ { p[k, a + x] , BiExSum[k] } ], {k, 0, 6} ]; (* p_k-sequence = { p_k(x) } *)
Print[ TableForm [T2] ]  (* Sequences Table Confirms: p_n(a+x)= BiExSum[n] *)
```

---------------------------- TABLE-T2 ------------------------------

$p_k(a+x)$ =	BiExSum[k]
1	1
$a + x$	$a + x$
$(a + x)^2$	$(a + x)^2$
$(a + x)(1 + a^2 + 2ax + x^2)$	$(a + x)(1 + a^2 + 2ax + x^2)$
$(a + x)^2(4 + a^2 + 2ax + x^2)$	$(a + x)^2(4 + a^2 + 2ax + x^2)$
$(a + x)(1 + a^2 + 2ax + x^2)(9 + a^2 + 2ax + x^2)$	$(a + x)(1 + a^2 + 2ax + x^2)(9 + a^2 + 2ax + x^2)$
$(a + x)^2(4 + a^2 + 2ax + x^2)(16 + a^2 + 2ax + x^2)$	$(a + x)^2(4 + a^2 + 2ax + x^2)(16 + a^2 + 2ax + x^2)$

(The above *Table 2 confirms* that *this Model* Satisfies the p-Binomial Identity for $0 \leq k \leq 6$).

Article 6. MODEL #06.

Features of this Model #06: (a)-- the *Operator L* was assigned the δ-sequence as the λ-coefficients; (b)--then, the *Operator Q* was assigned the δ-sequence as the μ-coefficients; (c)--for the (a)/(b) assignment cases, *Tables Illustrating both* p_k and q_k entries are displayed.

```
Clear[L, Q, p]
δ = {1, 1/2, 3/8, 5/16, 35/128, 63/256, 231/1024, 429/2048, 6435/32768, 12155/65536} ;
L_oper[d] = Sum[δ[[k]] d^k, {k, 1, 10}]  (*  L_operator polynomial profile  *)
Q_oper[d] = Series[d / L_oper[d], {d, 0, 10}]  (*  Q_operator polynomial profile  *)
```

$$d+\frac{d^2}{2}+\frac{3d^3}{8}+\frac{5d^4}{16}+\frac{35d^5}{128}+\frac{63d^6}{256}+\frac{231d^7}{1024}+\frac{429d^8}{2048}+\frac{6435d^9}{32768}+\frac{12155d^{10}}{65536}$$

(* L_profile display with δ-Coefficients *)

$$1-\frac{d}{2}-\frac{d^2}{8}-\frac{d^3}{16}-\frac{5d^4}{128}-\frac{7d^5}{256}-\frac{21d^6}{1024}-\frac{33d^7}{2048}-\frac{429d^8}{32768}-\frac{715d^9}{65536}+\frac{21879d^{10}}{131072}+O[d]^{11}$$ (* Q_profile display *)

```
L[f_] := Sum[δ[[k]] ∂_{x,k} f, {k, 1, 9}]   (*  L_delta Operator code  *)
Q[f_] := f - ∂_{t,1} f/2 - ∂_{t,2} f/8 - ∂_{t,3} f/16 - 5 ∂_{t,4} f/128 - 7 ∂_{t,5} f/256 - 21 ∂_{t,6} f/1024 - 33 ∂_{t,7} f/2048
         - 429 ∂_{t,8} f/32768 - 715 ∂_{t,9} f/65536   (*  Q Operator code: companion with L_delta *)

p[0, x_] := 1  (*  p0(x) declared as initial polynomial  *)
p[k_, x_] := Integrate[Q[k p[k - 1, t]], {t, 0, x}] (* p-Binomial Sequence  *)
Print["pk(x)", "                              ", "Lpk(x)  =  k pk-1(x)"]
T1 = Table[ Factor[{p[k, x], L[p[k, x]]}], {k, 0, 6}] ;
Print[TableForm[T1]]
```

(* ------------------ TABLE_T1 ---------------------------------*)

$p_k(x)$	$Lp_k(x) = k\, p_{k-1}(x)$
1	0
x	1
$(-1+x)\,x$	$2x$
$\frac{1}{4}x\,(3-12x+4x^2)$	$3\,(-1+x)\,x$
$x^2\,(6-6x+x^2)$	$x\,(3-12x+4x^2)$
$\frac{1}{16}x\,(-15-120x+360x^2-160x^3+16x^4)$	$5x^2\,(6-6x+x^2)$
$x^3\,(-60+60x-15x^2+x^3)$	$\frac{3}{8}x\,(-15-120x+360x^2-160x^3+16x^4)$

```
Clear[L, Q, q]
```

$$L[f\_] := \partial_{\{x,1\}} f - \frac{\partial_{\{x,2\}} f}{2} - \frac{\partial_{\{x,3\}} f}{8} - \frac{\partial_{\{x,4\}} f}{16} - \frac{5\,\partial_{\{x,5\}} f}{128} - \frac{7\,\partial_{\{x,6\}} f}{256} - \frac{21\,\partial_{\{x,7\}} f}{1024} - \frac{33\,\partial_{\{x,8\}} f}{2048} - \frac{429\,\partial_{\{x,9\}} f}{32768} - \frac{715\,\partial_{\{x,10\}} f}{65536} \quad \text{(* L Operator code: companion with Q\_delta *)}$$

$$Q[f\_] := \sum_{k=1}^{10} \delta[[k]]\, \partial_{\{t,k-1\}} f \quad \text{(* Q\_delta Operator code *)}$$

$$q[0, x\_] := 1 \quad \text{(* } q_0(x) \text{ declared as initial polynomial *)}$$

$$q[k\_, x\_] := \int_0^x Q[k\, q[k-1, t]]\, dt \quad \text{(* q-Binomial Sequence *)}$$

```
Print[ "qk(x)" , "                                  " , "Lqk(x)  =  k qk-1(x)"]
T2 = Table[  Factor[{q[k, x] , L[ q[k, x] ] } ], {k, 0, 6} ]  ;
Print[ TableForm[T2] ]
```

* ------------------ TABLE_T2 ---------------------- ------*)

$q_k(x)$	$Lq_k(x) = k\, q_{k-1}(x)$
1	0
x	1
$x\,(1+x)$	$2\,x$
$\frac{1}{4}\,x\,(15+12\,x+4\,x^2)$	$3\,x\,(1+x)$
$x\,(24+18\,x+6\,x^2+x^3)$	$x\,(15+12\,x+4\,x^2)$
$\frac{1}{16}\,x\,(3465+2520\,x+840\,x^2+160\,x^3+16\,x^4)$	$5\,x\,(24+18\,x+6\,x^2+x^3)$
$x\,(2520+1800\,x+600\,x^2+120\,x^3+15\,x^4+x^5)$	$\frac{3}{8}\,x\,(3465+2520\,x+840\,x^2+160\,x^3+16\,x^4)$

The below presentation *VERIFIES THAT the q-sequence* IS a *p-Binomial Sequence;*
hence: $p_n(a+x)$ = [p – Binomial Expansion Sum] (for $0 \leq n \leq 4$) .

$$BiExSum[N\_] := \sum_{k=0}^{N} Binomial[N, k]\; q[N-k, a]\; q[k, x] \quad \text{(* p-Binomial Expansion Sum *)}$$

```
Print[ "qk(a+x)" , "                    =                    " , "BiExSum[k]"]
T3 = Table[ Factor[ { q[k, a + x] , BiExSum[k] } ], {k, 0, 4} ]; (* qk-sequence = { qk(x) } *)
Print[ TableForm [T3] ]  (* Sequences Table Confirms: qn(a+x)= BiExSum[n]  *)
```

(* -------------------------TABLE_T3 -----------------------------*)

$q_k(a+x)$ =	$BiExSum[k]$
1	1
$a+x$	$a+x$
$(a+x)\,(1+a+x)$	$(a+x)\,(1+a+x)$
$\frac{1}{4}\,(a+x)\,(15+12\,a+4\,a^2+12\,x+8\,a\,x+4\,x^2)$	$\frac{1}{4}\,(a+x)\,(15+12\,a+4\,a^2+12\,x+8\,a\,x+4\,x^2)$
$(a+x)\,(24+18\,a+6\,a^2+a^3+18\,x+12\,a\,x+3\,a^2\,x+6\,x^2+3\,a\,x^2+x^3)$	$(a+x)\,(24+18\,a+6\,a^2+a^3+18\,x+12\,a\,x+3\,a^2\,x+6\,x^2+3\,a\,x^2+x^3)$