Lemmas on *Polynomial Operators*

(From: J B Barksdale, Jr || Profiles & Perspectives of p-Biomial Sequences)
(06/25/2018)

Introduction. Consider the following developmental details which render the below displayed *Lemmas* regarding *Polynomial (differential) Operators of Order n* defined on *subspaces*, $\mathbb{P}^n \subset \mathbb{P}^\infty$, of the vector space of polynomials over the real field. In the discussions which follow, $\mathbb{P}^n = \{$ subspace of real polynomials of *degree* $\leq n$ $\}$.

LEMMA #01. Given a *n^th^-order* $(\lambda_n \neq 0)$, *unit degree decreasing* $(\lambda_1 \neq 0)$ *polynomial (differential) operator L* defined by: $L = L(D) = \sum_{k=1}^{n} \lambda_k D^k$, then there exists a *m^th^-order* $(\mu_m \neq 0)$ *degree preservinging* $(\mu_0 \neq 0)$ *polynomial operator*, *Q(D)* $= \sum_{j=0}^{m} \mu_j D^j$, such that: $(LQ)p = Dp$ for each polynomial $p \in \mathbb{P}^{m+1}$.

Proof of LEMMA #01 . Let such *n^th^ order Operator L*, as above, be given. Noting the *order of ascending powers* of $L(D)$, repeatedly apply steps of the *Division Algorithm* as *described, detailed and presented* in Item (1.1), below. Note that the *DA application* is continued until a $\mu_m \neq 0$ is created, where $m = \inf(M)$, and $M = \{k \in \mathbb{N} \mid k \geq n \ \wedge \ \mu_k \neq 0 \}$. Then,

(1.1) $$\frac{D}{L} = \frac{D}{L(D)} = \frac{D}{\sum_{k=1}^{n} \lambda_k D^k} = \sum_{j=0}^{m} \mu_j D^j + \frac{R(D)}{L(D)} = Q(D) + \frac{R}{L},$$

which implies

(1.2) $$D = LQ + R(D),$$

where $R(D) = D^{m+2}\sigma(D)$, and σ denotes a polynomial of *degree* $(n-2)$. Note the *remainder, R(D),* does have *minimal order* $(m+2)$. Hence, from the above details, the *Division Algorithm* was executed until *quotient* Q did achieve *order* m, and so that the *remainder* R would have *minimal order* $(m+2)$. Therefore, $p \in \mathbb{P}^{m+1}$ implies

(1.3) $$Dp = (LQ + R)p = (LQ + R)p = (LQ)p, \quad \text{since } \mathbb{P}^{m+1} \subset \mathrm{Ker}(R). \quad \square$$

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<u>LEMMA #02</u>. Given a m^{th}*-order* ($\mu_m \neq 0$)*, degree preserving* ($\mu_0 \neq 0$) *polynomial (differential) operator* Q defined by: $Q = Q(D) = \sum_{j=0}^{m} \mu_j D^j$, then there exists a n^{th}*-order* ($\lambda_n \neq 0$)*, unit degree decreasing* ($\lambda_1 \neq 0$) *polynomial operator* L defined by: $L = L(D) = \sum_{k=1}^{n} \lambda_k D^k$ such that: $(LQ)p = Dp$ for each polynomial $p \in \mathbb{P}^n$.

<u>*Proof of LEMMA #02*</u> . Let such m^{th} *order Operator* Q, as above, be given. Noting the *order of ascending powers* of $Q(D)$, repeatedly apply steps of the *Division Algorithm* as *described, detailed and presented* in Item (2.1), below. Note that the *DA application* is continued until a $\lambda_n \neq 0$ is created, where $n = \inf(N)$, and $N = \{k \in \mathbb{N} \;\big|\; k \geq m \;\wedge\; \lambda_k \neq 0 \;\}$. Then,

(2.1) $$\frac{D}{Q} = \frac{D}{Q(D)} = \frac{D}{\sum_{j=0}^{m} \mu_j D^j} = \sum_{k=1}^{n} \lambda_k D^k + \frac{R(D)}{Q(D)} = L(D) + \frac{R}{Q}\,,$$

which implies

(2.2) $$D = QL + R(D)\,,$$

where $R(D) = D^{n+1}\sigma(D)$, and σ denotes a polynomial of *degree* $(m-1)$. Note the *remainder, **R(D)**,* does have *minimal order* $(n+1)$. Hence, from the above details, the *Division Algorithm* was executed until *quotient* L did achieve *order* n*,* and so that the *remainder* R would have *minimal order* $(n+1)$. Therefore, $p \in \mathbb{P}^n$ implies

(2.3) $$Dp = (QL + R)p = (QL)p, \quad \text{since } \mathbb{P}^n \subset \text{Ker}(R). \quad \square$$

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LEMMA #03. Given a n^{th}*-order* $(\lambda_n \neq 0)$, *unit degree decreasing* $(\lambda_1 \neq 0)$ *polynomial (differential) operator L* defined by: $L = L(D) = \sum_{k=1}^{n} \lambda_k D^k$,
then: $\mathbb{P}^m = L(\mathbb{P}^{m+1})$ for each finite subspace $\mathbb{P}^m \subset \mathbb{P}^\infty$.

Proof of LEMMA #03 . Let such n^{th} *order Operator* L, as above, be given.
Then: $p \in \mathbb{P}^{m+1} \implies (Lp) \in \mathbb{P}^m$. Hence, $L(\mathbb{P}^{m+1}) \subseteqq \mathbb{P}^m$. Now, let $\mathbb{B} = \{\beta_k\}_{k=0}^{k=m+1}$ denote a *basis* for $\mathbb{P}^{m+1}$. Since L is a *u.d.d operator*, then $\mathbb{S} = \{L\beta_k \mid 1 \leq k \leq m{+}1\}$ contains *one polynomial of each degree d*, $0 \leq d \leq m$. Consequently, $\mathbb{S}$ is a *basis* for $\mathbb{P}^m$.
Hence, $p \in \mathbb{P}^m \implies p = \sum_{k=1}^{m+1} c_k(L\beta_k) \; = \; \sum_{k=1}^{m+1} L(c_k\beta_k) \; = L\left(\sum_{k=1}^{m+1} c_k\beta_k\right)$, where $\sum_{k=1}^{m+1} c_k\beta_k \; = f \in \mathbb{P}^{m+1}$. Thus, $p \in \mathbb{P}^m \implies \exists\, f \in \mathbb{P}^{m+1} \;.\ni.\; p = Lf \in L(\mathbb{P}^{m+1})$.

The prior implication renders the inclusion $\mathbb{P}^m \subseteqq$ L($\mathbb{P}^{m+1}$), and the asserted equality $\mathbb{P}^m = L(\mathbb{P}^{m+1})$ is hereby established. ☐

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LEMMA #04. Given a n^{th}*-order* $(\lambda_n \neq 0)$, *unit degree decreasing* $(\lambda_1 \neq 0)$ *polynomial (differential) operator L* defined by: $L = L(D) = \sum_{k=1}^{n} \lambda_k D^k$, then there exists a *Unique* m^{th}*-Order* $(\mu_m \neq 0)$ *degree preservinging* $(\mu_0 \neq 0)$ *polynomial operator*, *Q(D)* $= \sum_{j=0}^{m} \mu_j D^j$, such that: $(LQ)p = Dp$ for each polynomial $p \in \mathbb{P}^{m+1}$.

Proof of LEMMA #04 . Let such n^{th} *order Operator L*, as above, be given. Now, applying *Lemma #01*, $\exists\, Q \;.\ni.\; p \in \mathbb{P}^{m+1} \Rightarrow (LQ)p = Dp$, where $Q = \sum_{j=0}^{m} \mu_j D^j$ and $\mu_0 \neq 0 \neq \mu_m$. In order to establish that Q is a *unique such operator*, suppose that *both Q and* $\overline{Q}$ satisfy the hypothesis of *Lemma #01*. Then, $p \in \mathbb{P}^{m+1} \implies$ $(LQ)p = Dp = (L\overline{Q})p$, where $\overline{Q} = \sum_{j=0}^{m} \overline{\mu}_j D^j$ and $\overline{\mu}_0 \neq 0 \neq \overline{\mu}_m$. Consequently, $(QL)p = (LQ)p = (L\overline{Q})p = (\overline{Q}L)p$, for all $p \in \mathbb{P}^{m+1}$. Now, from *Lemma #03*, recall that $L(\mathbb{P}^{m+1}) = \mathbb{P}^m$. Hence, $p \in \mathbb{P}^{m+1}$ implies that: $0 = (QL)p - (\overline{Q}L)p$. Hence, $0 = (Q - \overline{Q})[Lp] = (Q - \overline{Q})f$, where $f = (Lp) \in L(\mathbb{P}^{m+1}) = \mathbb{P}^m$. Consequently, $\mathbb{P}^m \subseteqq \mathrm{Ker}(Q - \overline{Q})$. Therefore, $\frac{x^m}{m!} \in \mathbb{P}^m \implies 0 = (Q - \overline{Q})[\frac{x^m}{m!}] = \sum_{j=0}^{m} (\mu_j - \overline{\mu}_j) D^j [\frac{x^m}{m!}]$

$$\implies 0 = \sum_{j=0}^{m} (\mu_j - \overline{\mu}_j)\, [\tfrac{x^{m-j}}{(m-j)!}] \implies 0 = (\mu_j - \overline{\mu}_j) \text{ , for } 0 \leq j \leq m,$$

$$\implies \mu_j = \overline{\mu}_j \text{ , for } 0 \leq j \leq m, \text{ by noting that } X = \left\{ \tfrac{x^{m-j}}{(m-j)!} \;\middle|\; 0 \leq j \leq m \right\}$$

is a *linearly independent set* by virtue of serving as a *basis* for $\mathbb{P}^m$. Accordingly, $Q = \overline{Q}$; thus, the demonstration asserting such *unique*, *companion operator for* a *given* L is hereby complete. ☐

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LEMMA #05. Given a n^{th}*-order* $(\lambda_n \neq 0)$, *unit degree decreasing* $(\lambda_1 \neq 0)$ *polynomial (differential) operator L* defined by: $L = L(D) = \sum_{k=1}^{n} \lambda_k D^k$, then there exists a *Unique* m^{th}*-Order* $(\mu_m \neq 0)$ *degree preservinging* $(\mu_0 \neq 0)$ *polynomial operator, Q(D)* $= \sum_{j=0}^{m} \mu_j D^j$, such that: $(LQ)p = Dp$ for each polynomial $p \in \mathbb{P}^{m+1}$. (Note: Viewing the *symbol D* as an *indeterminant,* the below referenced polynomial operators can be viewed as *elements of the Euclidean ring* E *of polynomials (* $\mathbb{P}^{\infty}$*) over the field of real numbers*).

Proof of LEMMA #05 . Let such n^{th} *order Operator L*, as above, be given. Noting the *order of ascending powers* of $L(D)$, repeatedly apply steps of the *Division Algorithm* as *described, detailed and presented* in Item (5.1), below. Note that *application of the DA* is continued until $\mu_m \neq 0$ is created, where $m = \inf(M)$, and $M = \{k \in \mathbb{N} \mid k \geq n \ \wedge \ \mu_k \neq 0 \ \}$. Then,

(5.1) $$\frac{D}{L} = \frac{D}{L(D)} = \frac{D}{\sum_{k=1}^{n} \lambda_k D^k} = \sum_{j=0}^{m} \mu_j D^j + \frac{R(D)}{L(D)} = Q(D) + \frac{R}{L},$$

which implies

(5.2) $$D = LQ + R(D) = LQ + R,$$

where $R(D) = D^{m+2}\sigma(D)$, and σ denotes a polynomial of *degree* $(n{-}2)$. Note the *remainder, R(D),* does have *minimal order* $(m{+}2)$. Hence, from the above details, the *Division Algorithm* was executed to construct *the quotient* Q and *the remainder R* so that Q *achieved order* $m = \inf(M)$, and R *consequently*, *achieved order* $(n{+}m)$, and with *minimal order* $(m{+}2)$. Hence, via *the application of the Division Algorithm* to the *polynomial ring elements D,* $L \in E$ belonging to the *Euclidean Ring E,* ring elements Q *and R* were constructed via the *Division Algorithm* for *Euclidean Rings (Domains).* Such ring (domain) elements are thus related as displayed in *Item (5.2).* Hence,

(5.3) $LQ = (D - R)$, via the *Division Algorithm.*

From *Item (5.3)* and the *unique factorization property* for *Euclidean ring elements,* the *operators L,Q* $\in$ *E* constitute a *unique pair of factors* as a consequence of the *Division Algortithm* applied to *given Euclidean ring elements L,D* $\in$ *E.*

Hence, from the above details, the *Division Algorithm* was executed until *quotient* Q did achieve *order* m, and so that the *remainder* R would have *minimal order* $(m{+}2)$. Therefore, $p \in \mathbb{P}^{m+1}$ implies

(5.4) $Dp = (LQ + R)p = (LQ)p$, since $\mathbb{P}^{m+1} \subset \mathrm{Ker}(R)$. □