Profiles & Perspectives of *p-Binomial Sequences*

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Article 00: p-Binomial Sequence Necessary Properties. Suppose that the following *basis* $\mathbb{B} \subset \mathbb{P}^N$ for the subspace $\mathbb{P}^N$ of the space of polynomials $\mathbb{P}^\infty$ over the real number field $\mathbb{R}$ satifies *Item (0.1),* below. Hence, given such basis $\mathbb{B} = \{p_k(x) \,\big|\, \deg(p_k) = k\,,\, 0 \le k \le N\}$, suppose for each n, $0 \le n \le N, \forall\, a \in \mathbb{R}$, such basis satisfies the *p-Binomial Expansion Identity (0.1),*

(0.1) $\quad p_n(a+x) = \sum_{k=0}^{n} \binom{n}{k} p_{n-k}(a)\, p_k(x)$. Then, *basis* $\mathbb{B}$ and $a = 0$

$$\Longrightarrow\ p_N(x) = \sum_{m=0}^{N} \binom{N}{m} p_{N-m}(0)\, p_m(x) \ \Longrightarrow\ p_0(0) = 1,\ p_{N-m}(0) = 0 \text{ for } 1 \le m < N.$$

Consequenly, a *necessary condition* for such *basis* $\mathbb{B}$ to satisfy *Item (0.1)* is that the *constant basis element* $p_0(x) = 1$ and that $p_k(0) = 0$ for all others $(k > 0)$. Thus, such bases $\mathbb{B}$ must necessarily be described as in *Item (0.2).*

(0.2) $\quad \mathbb{B} = \{p_k(x) \,\big|\, p_0(x) \equiv 1,\ \deg(p_k) = k \text{ and } p_k(0) = 0\ ,\ 0 < k \le N\}$.

Regarding the following developments, such bases as in *Item (0.2)* shall be referenced as *simple bases.*

Article 01: A p-Binomial Sequence Model. Imagine the following *basis* $\mathbb{B} \subset \mathbb{P}^N$ for the subspace $\mathbb{P}^N$ of space of polynomials $\mathbb{P}^\infty$ over the real number field $\Re$. Hence, consider *simple basis* $\mathbb{B} = \{\beta_k(x) \,\big|\, \beta_k(x) = x^k\}$ for $\mathbb{P}^N$ detailed by *Item (1.1).*

(1.1) $\quad \mathbb{B} = \{\beta_k(x) \,\big|\, \beta_0(x) \equiv 1,\ \deg(\beta_k) = k \text{ and } \beta_k(0) = 0\ ,\ 0 < k \le N\}$.

Note the following properties of such detailed, simple basis elements of $\mathbb{B}$.

(1.2) $\quad \beta_k(0) = 0$ for $k \ge 1$, and $\beta_k^{(m)}(0) = D^m \beta_k(x)\Big|_{x=0} = \frac{k!}{(k-m)!}\, \beta_{k-m}(0)$.

Thus,

(1.3) $\quad \beta_k(0) = 0$ for $k \ge 1$, and so $\beta_{k-m}(0) = \delta_k^m = \begin{cases} 1, & \text{if } k = m \\ 0, & \text{otherwise} \end{cases}$.

Then, for each n , $0 \le n \le N$, there exists scalars $\{c_k\}_{k=0}^{k=n}$ such that

(1.4) $\beta_n(a+x) = (a+x)^n = \sum_{k=0}^{n} c_k \beta_k(x)$, since $(a+x)^n \in \mathbb{P}^N$.

Now, observe that for each m, $0 \le m \le n$, $D^m(a+x)^n = \sum_{k=0}^{n} c_k D^m \beta_k(x)$

$\implies \frac{n!}{(n-m)!}(a+x)^{n-m} = \sum_{k=0}^{n} c_k \frac{k!}{(k-m)!} \beta_{k-m}(x)$; evaluating at $x=0$,

(1.5) $\frac{n!}{(n-m)!} a^{n-m} = \sum_{k=0}^{n} c_k \frac{k!}{(k-m)!} \beta_{k-m}(0) = \sum_{k=0}^{n} c_k \frac{k!}{(k-m)!} \delta_k^m$

$\implies \frac{n!}{(n-m)!} a^{n-m} = c_m \frac{m!}{0!} = m!\, c_m \implies c_m = \frac{n!}{(n-m)!\, m!} a^{n-m}$.

Consequently, substituting the results of *Item (1.5)* into *Item (1.4)* renders
Item (1.6), below:

$$\boxed{(a+x)^n = \beta_n(a+x) = \sum_{m=0}^{n} \frac{n!}{(n-m)!\, m!} \beta_{n-m}(a)\, \beta_m(x) = \sum_{m=0}^{n} \binom{n}{m} a^{n-m} x^m} .$$