**Euclidean Space** \((n\text{-dimensional space})\): For each positive integer \(n\), we let \(R^n\) denote the set of all ordered \(n\)-tuples of the form

\[
    u = \{u_1, \ldots, u_n\},
\]

where \(u_i \in \mathbb{R}\) for \(1 \leq i \leq n\). Elements in \(R^n\) are called vectors. However for \(n = 1\), we simply refer to them as real numbers or scalars and write \(\mathbb{R}\) rather than \(R^1\).

Henceforth, let \(u = \{u_1, \ldots, u_n\} \) and \(v = \{v_1, \ldots, v_n\} \) be vectors in \(R^n\).

**0. Sum and Scalar Multiplication:** (i) We define the sum of vectors \(u\) and \(v\) by

\[
    u + v = \{u_1 + v_1, \ldots, u_n + v_n\}.
\]

(ii) For \(c \in \mathbb{R}\), the scalar product \(c u\) is defined by \(c u = \{c u_1, \ldots, c u_n\}\).

We can then define the difference \(v - u\) by

\[
    v - u = v + (-1)u = \{v_1 - u_1, \ldots, v_n - u_n\},
\]

which gives the directional vector from \(u\) to \(v\).

With these operations of addition and scalar multiplication, \(R^n\) is a vector space over the field of real numbers.

**Properties of Addition**

1. If \(u, v \in R^n\), then \(u + v \in R^n\). (closure)
2. \((u + v) + w = u + (v + w)\) for all \(u, v, w \in R^n\). (associative)
3. \(u + v = v + u\) for all \(u, v \in R^n\). (commutative)
4. There exists a “0” element in \(R^n\), denoted \(\vec{0}\), such that \(\vec{0} + v = v + \vec{0} = v\) for all \(v \in R^n\). (additive identity) Specifically, \(\vec{0} = \{0, \ldots, 0\}\), which we call the zero vector.
5. For every \(v \in R^n\), there exists an element \(-v\) such that \(v + (-v) = \vec{0} = -v + v\) (additive inverse) Specifically, \(-\{v_1, \ldots, v_n\} = \{-v_1, \ldots, -v_n\}\).

**Properties of Scalar Multiplication**

1. If \(v \in R^n\) and \(c\) is any scalar (i.e., real number), then \(cv \in R^n\). (closure)
2. \((cd)v = (cd)v\) for all \(v \in R^n\) and all \(c, d \in \mathbb{R}\). (associative)
3. \(c(u + v) = cu + cv\) for all \(u, v \in R^n\) and all \(c \in \mathbb{R}\). (scalar distributive)
4. \((c + d)v = cv + dv\) for all \(v \in R^n\) and all \(c, d \in \mathbb{R}\). (vector distributive)
5. For the scalar 1, \(1v = v\) for all \(v \in R^n\). (multiplicative identity)
I. Norm: The norm (or length or magnitude) of a vector \( u \in \mathbb{R}^n \) is defined by

\[
\|u\| = \sqrt{u_1^2 + \ldots + u_n^2}.
\]

In \( \mathbb{R}^2 \), the norm of the vector \( u = (x, y) \) is simply the length of the segment from the origin \((0, 0)\) to the point \((x, y)\), and is given by \( \sqrt{x^2 + y^2} \).

Properties of Norm

(i) \( \|u\| \geq 0 \), and \( \|u\| = 0 \) if and only if \( u = \mathbf{0} \);  
(ii) \( \|u\|^2 = u_1^2 + \ldots + u_n^2 \);  
(iii) \( \|c u\| = |c| \|u\| \) for every scalar \( c \in \mathbb{R} \).

II. Distance: The distance \( d(u,v) \) between vectors \( u \) and \( v \) is given by the length of the vector from \( u \) to \( v \):

\[
d(u,v) = \|v - u\| = \sqrt{(v_1 - u_1)^2 + \ldots + (v_n - u_n)^2}.
\]

Properties of Distance

For all for all \( u, v \in \mathbb{R}^n \):

(i) \( d(u,v) \geq 0 \)  
(ii) \( d(u,u) = 0 \) and \( d(u,v) = 0 \) if and only if \( u = v \)  
(iii) \( d(u,v) = d(v,u) \).

III. Dot Product: We define the dot product \( u \cdot v \) (also called inner product) between vectors \( u \) and \( v \) in \( \mathbb{R}^n \) by \( u \cdot v = u_1 v_1 + \ldots + u_n v_n \).

Properties of Dot Product

(a) \( u \cdot u = u_1^2 + \ldots + u_n^2 = \|u\|^2 \geq 0 \).  
(b) \( \|u\| = \sqrt{u \cdot u} \)  
(c) \( u \cdot v = v \cdot u \)  
(d) For any scalar \( c \), \((c u) \cdot v = c \times (u \cdot v)\) and \( u \cdot (c v) = c \times (u \cdot v)\).  
(e) For another vector \( w \), \((u + v) \cdot w = (u \cdot w) + (v \cdot w)\).

Theorem (Cauchy-Schwarz Inequality). Let \( u \) and \( v \) be vectors in \( \mathbb{R}^n \). Then

\[
\|u \cdot v\| \leq \|u\| \|v\|.
\]
Proof. Let \( k \) be a scalar and consider the vector \( ku + v \). Then

\[
0 \leq \| ku + v \|^2 = (ku + v) \cdot (ku + v) \\
= k^2 (u \cdot u) + k (u \cdot v) + k (v \cdot u) + (v \cdot v) \\
= \| u \|^2 k^2 + 2 (u \cdot v) k + \| v \|^2.
\]

This expression defines a quadratic in \( k \) which is always non-negative; so the quadratic has zero roots or just one root. Thus the discriminant \( "b^2 - 4ac" \) must be less than or equal to 0, where \( a = \| u \|^2 \), \( b = 2(u \cdot v) \), and \( c = \| v \|^2 \). (If the discriminant were positive, then the quadratic would have two roots and therefore would have to be negative over some interval.)

Thus, \( b^2 - 4ac = 4(u \cdot v)^2 - 4\| u \|^2 \| v \|^2 \leq 0 \), which implies that \( (u \cdot v)^2 \leq \| u \|^2 \| v \|^2 \). By taking square roots we obtain \( |u \cdot v| \leq \| u \| \| v \| \). QED

**Theorem** (Triangle Inequality). Let \( u \) and \( v \) be vectors in \( \mathbb{R}^n \). Then

\[
\| u + v \| \leq \| u \| + \| v \|.
\]

Proof: We simply expand \( \| u + v \|^2 \). The first inequality below arises from the fact that the number \( u \cdot v \) is less than or equal to its absolute value. The second inequality is from Cauchy-Schwarz:

\[
\| u + v \|^2 = (u + v) \cdot (u + v) = \| u \|^2 + \| v \|^2 + 2(u \cdot v) \\
\leq \| u \|^2 + \| v \|^2 + 2 |u \cdot v| \\
\leq \| u \|^2 + \| v \|^2 + 2 \| u \| \| v \| \\
= (\| u \| + \| v \|)^2.
\]

Because the square root function is strictly increasing, we maintain the inequality by taking square roots. Thus, we obtain \( \| u + v \| \leq \| u \| + \| v \| \). QED

**Note:** For vectors \( u \) and \( v \) in \( \mathbb{R}^2 \), the triangle inequality states that length of the diagonal \( u + v \) can be no more than the sum of the lengths of the two sides \( u \) and \( v \) (hence, the name triangle inequality). The result can be generalized to the distance between vectors as shown next.
Shortest Distance Between Points

Theorem. Let $u$ and $v$ be vectors in $\mathbb{R}^n$. For any other vector $w$, we have $d(u,v) \leq d(u,w) + d(w,v)$.

Proof. We use the definition of distance and apply the triangle inequality:

$$d(u,v) = \| v - u \| = \| (v - w) + (w - u) \| \leq \| v - w \| + \| w - u \| = d(w,v) + d(u,w) = d(u,w) + d(w,v).$$

QED

The result states that the distance directly from $u$ to $v$ is less than or equal to the sum of the distances from $u$ to $w$ then from $w$ to $v$. We note that if $u$, $v$, and $w$ are collinear with $w$ between $u$ and $v$, then $d(u,v)$ will equal $d(u,w) + d(w,v)$.

Absolute Value

For $x \in \mathbb{R}$, we define the absolute value of $x$ by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$$

We shall show that $|x|$ is really the norm $\| x \|$ when considering $x$ to be an element of the Euclidean space $\mathbb{R} = \mathbb{R}^1$. So we will be able to apply all the properties of norm and distance to the absolute value function.

Properties of Absolute Value

(i) For all $x \in \mathbb{R}$, $|x| \geq 0$. If $x \geq 0$, then $|x| = x \geq 0$. If $x < 0$, then $|x| = -x > 0$.

(ii) For all $x \in \mathbb{R}$, $x \leq |x|$. If $x \geq 0$, then $x = |x|$. If $x < 0$, then $x < 0 \leq |x|$.

(iii) $|x| = 0$ if and only if $x = 0$. That is, $|0| = 0$ and $|x| > 0$ for $x \neq 0$.

(iv) For all $x \in \mathbb{R}$, $\sqrt{x^2} = |x|$.

If $x \geq 0$, then $|x| = x = \sqrt{x^2}$. If $x < 0$, then $\sqrt{x^2} = \sqrt{(-x)(-x)} = (-x) = |x|$.

From (iv), we see that $\| x \| = \sqrt{x^2} = |x|$; thus, the Euclidean norm of $\mathbb{R}^1$ is the same as the absolute value in $\mathbb{R}$. 
(v) Cauchy-Schwarz: For all \( x, y \in \mathbb{R}, \) \( |xy| = |x||y|. \) Consider three cases. (i) If \( x \geq 0 \) and \( y \geq 0, \) then \( xy \geq 0 \) and thus \( |xy| = xy = |x||y|. \) (ii) If \( x < 0 \) and \( y < 0, \) then \( xy > 0 \) and \( |xy| = xy = (-1)x(-1)y = |x||y|. \) (iii) If one of \( x, y \) is negative and the other non-negative, then \( xy \leq 0; \) thus, \( |xy| = -(xy) = |x||y|. \)

(vi) \( |-x| = |x| \) Here we apply (iv) to obtain \( |-x| = |-1x| = |-1||x| = |x| = |x|. \)

(vii) \( |x|^2 = x^2 \) Here we have \( |x|^2 = \begin{cases} x^2 & \text{if } x \geq 0 \\ (-x)^2 & \text{if } x < 0 \end{cases} = x^2 \) for all \( x. \)

(viii) Triangle Inequality for Absolute Value: For all \( x, y \in \mathbb{R}, \)
\[
|x + y| \leq |x| + |y|.
\]

A direct proof in \( \mathbb{R} \) is easier than the general proof in \( \mathbb{R}^n. \) For all \( x, y \in \mathbb{R}, \)
\[
|x + y|^2 = (x + y)^2 = x^2 + 2xy + y^2 \leq x^2 + 2|xy| + y^2 = |x|^2 + 2|x||y| + |y|^2 = (|x| + |y|)^2.
\]

Because the square root function is strictly increasing, we maintain the inequality by taking square roots. Thus, we obtain \( |x + y| \leq |x| + |y|. \)

(ix) Distance in \( \mathbb{R}: \) For all \( x, y \in \mathbb{R}, \) we define the distance between \( x \) and \( y \) by
\[
d(x, y) = |x - y|
\]

We shall use the triangle inequality repeatedly in \( \mathbb{R} \) as
\[
|x - y| = |x - z + z - y| \leq |x - z| + |z - y|.
\]
That is, \( d(x, y) \leq d(x, z) + d(z, y). \)

**Theorem.** For all \( x \in \mathbb{R}, \) \( |x^n| = |x|^n \) for all integers \( n \geq 1. \)

**Proof.** We shall use induction on \( n. \) For \( n = 1, \) we have \( |x^1| = |x| = |x|^1 \) for all \( x \in \mathbb{R}. \)

For \( n = 2, \) we have independently shown that \( |x^2| = x^2. \) And because \( x^2 \geq 0, \) we then have \( |x|^2 = x^2 = |x^2| \) for all \( x \in \mathbb{R}. \)

Now assume that for all \( x \in \mathbb{R}, \) \( |x^n| = |x|^n \) for some particular \( n \geq 1. \) Then
\[
|x^{n+1}| = |x^n x| = |x^n||x| = |x|^n |x| = |x|^{n+1}, \text{ for all } x \in \mathbb{R}.
\]

By mathematical induction, the result holds for all integers \( n \geq 1. \)
Exercises

1. Prove that for all integers \( n \geq 1 \) and all real numbers \( x_1, x_2, \ldots, x_n \)
   \[(a) \ |x_1 + x_2 + \ldots + x_n| \leq |x_1| + |x_2| + \ldots + |x_n| \]
   \[(b) \ |x_1 \times x_2 \times \ldots \times x_n| = |x_1| \times |x_2| \times \ldots \times |x_n| . \]

2. For all real numbers \( x \) and \( y \), prove that
   \[||x| - |y|| \leq |x - y| . \]

3. Let \( u = \{u_1, \ldots, u_n\} \) and \( v = \{v_1, \ldots, v_n\} \) be vectors in \( R^n \). Consider the individual coordinates as points in \( \mathbb{R} \). Prove that for \( 1 \leq i \leq n \),
   \[(a) \ |u_i| \leq \|u\| \quad \quad (b) \ d(u_i, v_i) \leq d(u, v) . \]

4. Prove the following results about the absolute value:
   \[(a) \ |x - y| = 0 \text{ if and only if } x = y . \]
   \[(b) \ |x - y| = |y - x| \]

5. Suppose that \( |x - y| < \varepsilon \) for all \( \varepsilon > 0 \). Prove that \( x = y \).