Let $Z \sim N(0, 1)$ be the standard normal distribution having mean 0 and standard deviation 1. The $z$-score for probability $r$ is the value $z$ such that $P(-z \leq Z \leq z) = r$. That is, there is probability $r$ between the points $-z$ and $+z$.

We denote the total remaining outer probability by the symbol $\alpha$, where $r = 1 - \alpha$. Then there is $\alpha / 2$ probability at each tail. In this case, the $z$-scores are usually denoted by $z_{\alpha/2}$, where $\alpha / 2$ refers to the right-tail probability. Here are the most commonly used $z$-scores:

<table>
<thead>
<tr>
<th>$r$</th>
<th>$\alpha / 2$</th>
<th>$z_{\alpha/2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.90</td>
<td>0.05</td>
<td>1.645</td>
</tr>
<tr>
<td>0.95</td>
<td>0.025</td>
<td>1.96</td>
</tr>
<tr>
<td>0.98</td>
<td>0.01</td>
<td>2.326</td>
</tr>
<tr>
<td>0.99</td>
<td>0.005</td>
<td>2.576</td>
</tr>
</tbody>
</table>

Note: Any other $z$-score can be found with the built-in `invNorm` command from the DISTR menu. For example, if we want the 80% $z$-score, then there is 80% probability in the middle region. So there is $\alpha / 2 = 10\%$ at each tail. The (positive) $z$-score then comes from finding the inverse of 90% cumulative probability of the standard normal distribution and is approximately $z_{0.10} = 1.28$.

- Fact: For a general normal distribution, $X \sim N(\mu, \sigma)$, the bounds that contain inner probability $r = 1 - \alpha$ are given by

$$\mu \pm z_{\alpha/2} \sigma.$$
**Example.** Heights of adult women are normally distributed with a mean of 65.5 inches and a standard deviation of about 3 inches. Use \( z \)-scores to solve the following:

(a) What heights are such that 90% of all women are between these heights?
(b) What heights are such that 95% of all women are between these heights?
(c) What heights are such that 75% of all women are between these heights?

**Solution.** (a) The bounds that contain 90% of (normally distributed) measurements are \( \mu \pm 1.645\sigma \), which here are 65.5 \( \pm 1.645 \times 3 \). Thus, 90% of women are from 60.565 to 70.435 inches in height.

(b) The bounds that contain 95% of the measurements are \( \mu \pm 1.96\sigma \), which here are 65.5 \( \pm 1.96 \times 3 \). Thus, 95% of women are from 59.62 to 71.38 inches in height.

(c) First, we need the 75% \( z \)-score. If there is 75% inner probability, then there is 12.5% at each tail. Hence, the (positive) \( z \)-score comes from finding the inverse of 87.5% cumulative probability of the standard normal distribution and is about 1.15.

\[
\text{invNorm}(0.875, 0, 1) = 1.150349379
\]

So the bounds that contain 75% of the measurements are \( \mu \pm 1.15\sigma \), which here are 65.5 \( \pm 1.15 \times 3 \). Thus, 75% of women are from 62.05 to 68.95 inches in height.

**Exercise**

IQ scores are generally found to be normally distributed with a mean of 100 and a standard deviation of 15. Use \( z \)-scores to find the bounds that contain

(a) 95% of all scores
(b) 98% of all scores
(c) 40% of all scores
(b) 94% of all scores

**Answers**

(a) \( \mu \pm 1.96 \sigma \rightarrow 100 \pm 1.96 \times 15 \), or 70.6 to 129.4
(b) \( \mu \pm 2.326 \sigma \rightarrow 100 \pm 2.326 \times 15 \), or 65.11 to 134.89
(c) \( \mu \pm 0.5244 \sigma \rightarrow 100 \pm 0.5244 \times 15 \), or 92.134 to 107.866 (Use \text{invNorm}(0.7, 0, 1)\)
(d) \( \mu \pm 1.88 \sigma \rightarrow 100 \pm 1.88 \times 15 \), or 71.8 to 128.2 (Use \text{invNorm}(0.97, 0, 1)\)