A quadratic function has the form

\[ f(x) = ax^2 + bx + c \]

where \( a \neq 0 \). The graph of the function is a parabola that opens upward if \( a > 0 \), and opens downward for \( a < 0 \).

Given such a parabolic function, we will

(a) Find the vertex.  (b) State the \( y \)-intercept.  (c) Find the \( x \)-intercepts.

(d) Graph the function.  (e) Solve the equation \( f(x) = d \).

(f) Use the solution from (e) to solve the inequalities \( y \leq d \), \( y < d \), \( y \geq d \), or \( y > d \).

As an example throughout, we will use the function

\[ y = 3x^2 - 12x + 6 \]

The Vertex

The vertex \((x, y)\) is the point at which the graph changes from decreasing to increasing. The \( x \)-coordinate of the vertex is given by

\[ x = \frac{-b}{2a} \]

We then substitute this \( x \)-value into the function to find the \( y \)-coordinate of the vertex.
For \( y = 3x^2 - 12x + 6 \), the \( x \)-coordinate of the vertex is

\[
x = \frac{-b}{2a} = \frac{-(12)}{2(3)} = \frac{12}{6} = 2.
\]

The \( y \)-coordinate is then \( 3(2)^2 - 12(2) + 6 = -6 \). So the vertex is the point \((2, -6)\).

**The \( y \)-intercept**

The \( y \)-intercept of the function \( y = ax^2 + bx + c \) is always given by the constant term \( c \). That is, when \( x = 0 \), then \( y = c \). For \( y = 3x^2 - 12x + 6 \), the \( y \)-intercept is 6.

**The \( x \)-intercepts**

To find the \( x \)-intercepts of the quadratic \( y = ax^2 + bx + c \), we set \( y = 0 \) and solve for \( x \). That is, we must solve the quadratic equation

\[
ax^2 + bx + c = 0.
\]

If the quadratic does not factor, then we can use the quadratic formula

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\]

**Note:** The expression under the radical \( b^2 - 4ac \) is called the “discriminant” which determines how many solutions there are to the equation \( ax^2 + bx + c = 0 \).

(i) If \( b^2 - 4ac > 0 \), then there are two \( x \)-intercepts.
(ii) If \( b^2 - 4ac = 0 \), then there is one \( x \)-intercept.
(iii) If \( b^2 - 4ac < 0 \), then there are no solutions to \( ax^2 + bx + c = 0 \) and no \( x \)-intercepts.
(iv) If \( b^2 - 4ac \) is a perfect square, then the quadratic \( ax^2 + bx + c \) will factor.

<table>
<thead>
<tr>
<th>( y )</th>
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<tbody>
<tr>
<td>( b^2 - 4ac &gt; 0 )</td>
<td>( b^2 - 4ac = 0 )</td>
<td>( b^2 - 4ac &lt; 0 )</td>
</tr>
<tr>
<td>Two ( x )-intercepts</td>
<td>One ( x )-intercept</td>
<td>No ( x )-intercepts</td>
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</table>
Consider \( y = 2x^2 - x - 10 \). The discriminant is \( b^2 - 4ac = (-1)^2 - 4(2)(-10) = 1 + 80 = 81 \), which is positive and a perfect square. So there are two \( x \)-intercepts that can be found by factoring: \( 2x^2 - x - 10 = (2x - 5)(x + 2) = 0 \) \( \rightarrow 2x - 5 = 0 \) or \( x + 2 = 0 \). Thus,

\[
x = \frac{5}{2} \quad \text{and} \quad x = -2
\]

are the \( x \)-intercepts.

We now will find the \( x \)-intercepts of \( y = 3x^2 - 12x + 6 \) by using the quadratic formula:

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

\[
= \frac{-(-12) \pm \sqrt{(-12)^2 - 4(3)(6)}}{2(3)}
\]

\[
= \frac{12 \pm \sqrt{144 - 72}}{6} = \frac{12 \pm \sqrt{72}}{6} = 2 \pm \frac{6 \sqrt{2}}{6} = 2 \pm \sqrt{2}
\]

The approximate decimal values are \( 2 - \sqrt{2} \approx 0.585786 \) and \( 2 + \sqrt{2} \approx 3.4142 \).

**Graphing**

Once we have found the vertex, the \( y \)-intercept, and the \( x \)-intercepts, we can graph the parabola by plotting these points.

For \( y = 3x^2 - 12x + 6 \), the vertex is (2, –6), the \( y \)-intercept is 6 and the \( x \)-intercepts are \( 2 \pm \sqrt{2} \), which are about 0.585786 and 3.4142.
## Solving Other Equations

Given the quadratic function \( f(x) = ax^2 + bx + c \), we also can solve for the \( x \) that makes \( f(x) = d \). That is, we can solve the equation \( ax^2 + bx + c = d \). To do so, always set the equation equal to 0 by subtracting the \( d \) term, and then either factor or use the quadratic formula to solve the resulting equation.

Consider again the function \( f(x) = 3x^2 - 12x + 6 \). When does \( y = 21 \)?

Here we must solve the equation \( 3x^2 - 12x + 6 = 21 \). By subtracting 21 we obtain

\[
3x^2 - 12x - 15 = 0
\]

If we divide by 3, we obtain \( x^2 - 4x - 5 = 0 \) which factors as \((x - 5)(x + 1) = 0\). So the solutions are \( x = -1 \) and \( x = 5 \).

![Graph of the quadratic function with points (-1, 21) and (5, 21)](image)

## Solving Inequalities

Using the solution from above, solve the inequalities \( y \leq 21 \), \( y < 21 \), \( y > 21 \), and \( y \geq 21 \). From the graph, we see that

<table>
<thead>
<tr>
<th>Inequality</th>
<th>Solution</th>
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<tbody>
<tr>
<td>( y \leq 21 )</td>
<td>(-1 \leq x \leq 5) i.e., for ( x ) in ([-1, 5])</td>
</tr>
<tr>
<td>( y &lt; 21 )</td>
<td>(-1 &lt; x &lt; 5) i.e., for ( x ) in ((−1, 5))</td>
</tr>
<tr>
<td>( y &gt; 21 )</td>
<td>( x &lt; -1 ) or ( x &gt; 5 ) i.e., for ( x ) in ((−∞, -1) \cup (5, ∞))</td>
</tr>
<tr>
<td>( y \geq 21 )</td>
<td>( x \leq -1 ) or ( x \geq 5 ) i.e., for ( x ) in ((−∞, -1] \cup [5, ∞))</td>
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</tbody>
</table>
Exercises

1. Let  \( y = 3x^2 - 4x - 6 \).
   (a) Find the vertex.
   (b) Find the \( y \)-intercept.
   (c) Find the \( x \)-intercepts.
   (d) Graph. Show the intercepts and vertex.
   (e) Solve the equation \( y = -2 \) by factoring.
   (f) For which \( x \) is \( y \leq -2 \)? For which \( x \) is \( y > -2 \)?

2. Let  \( y = -2x^2 + 12x - 10 \).
   (a) Find the vertex.
   (b) Find the \( y \)-intercept.
   (c) Find the \( x \)-intercepts.
   (d) Graph. Show the intercepts and vertex.
   (e) Solve the equation \( y = 6 \) by factoring.
   (f) For which \( x \) is \( y \geq 6 \)? For which \( x \) is \( y < 6 \)?
Solutions

1. (a) The $x$-coordinate of the vertex is \( x = \frac{-b}{2a} = \frac{-(4)}{2 \times 3} = \frac{4}{6} = \frac{2}{3} \). The $y$-coordinate is \( 3\left(\frac{2}{3}\right)^2 - 4\left(\frac{2}{3}\right) - 6 = -\frac{22}{3} \). So the vertex is \( \left(\frac{2}{3}, -\frac{22}{3}\right) \).

(b) When $x = 0$, the $y$-intercept is $y = -6$.

(c) To find the $x$-intercepts, we will solve $3x^2 - 4x - 6 = 0$ with the quadratic formula. For our equation, $a = 3$, $b = -4$, and $c = -6$; hence, the solutions are:

\[
\begin{align*}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(4) \pm \sqrt{(-4)^2 - 4(3)(-6)}}{2(3)} \\
&= \frac{4 \pm \sqrt{88}}{6} = \frac{4 \pm 2\sqrt{22}}{6} = \frac{2 \pm \sqrt{22}}{3}
\end{align*}
\]

The approximate numerical solutions are $x \approx 2.23$ and $x \approx -0.8968$.

(e) Now solve $3x^2 - 4x - 6 = -2$, which is equivalent to $3x^2 - 4x - 4 = 0$. We can factor as $(3x + 2)(x - 2) = 0$. So $(3x + 2) = 0$ or $(x - 2) = 0$. The solutions to $y = 2$ are then $x = -2/3$ and $x = 2$.

(f) When $-2/3 \leq x \leq 2$, then $y \leq -2$. When $x < -2/3$ or $x > 2$, then $y > -2$. 
2. \( y = -2x^2 + 12x - 10 \)  
   (a) The \( x \)-coordinate of the vertex is \( x = \frac{-b}{2a} = \frac{-12}{2(-2)} = 3 \). The \( y \)-coordinate is \(-2(3)^2 + 12(3) - 10 = 8\) So the vertex is \((3, 8)\).  
   (b) When \( x = 0 \), we obtain \( y = -10 \) for the \( y \)-intercept.  
   (c) To find the \( x \)-intercepts, we must solve \(-2x^2 + 12x - 10 = 0\). Dividing by \(-2\), we instead can solve \(x^2 - 6x + 5 = 0\), which factors as \((x - 5)(x - 1) = 0\), so \(x = 1\) and \(x = 5\) are the \( x \)-intercepts. 
   Or, using the quadratic formula on the original equation \(-2x^2 + 12x - 10 = 0\), we obtain
   \[
   x = \frac{-12 \pm \sqrt{12^2 - 4(-2)(-10)}}{2(-2)} \\
   = \frac{-12 \pm \sqrt{64}}{-4} = \frac{-12 \pm 8}{-4} = 3 \pm 2
   \]
   (d) Now we must solve \(-2x^2 + 12x - 10 = 6\), which is equivalent to \(-2x^2 + 12x - 16 = 0\). Dividing by \(-2\), we have \(x^2 - 6x + 8 = 0\), which factors as \((x - 4)(x - 2) = 0\). Thus, \(x = 4\) and \(x = 2\) are the solutions to \(y = 6\).  
   (e) We have \(y \geq 6\) when \(2 \leq x \leq 4\). And \(y < 6\) when \(x < 2\) or \(x > 4\).