As an application of quadratic functions, we will consider the height of an object that moves straight upward or straight downward and is acted on only by gravity. There are three general cases that we will consider:

I. Drop an object from a specified height.

(a) Determine the time it takes to fall a certain distance and the speed at this time.
(b) Determine the time it takes to hit the ground and the impact speed.

II. Toss an object straight upward from a specified height with a specified initial velocity.

(a) Determine the maximum height.
(b) Determine the times at which the object attains a particular speed.
(c) Determine the total time it takes to return and hit the ground.
(d) Determine the impact speed upon hitting the ground.

III. An object springs from the ground and reaches a certain height.

(a) Determine the time it takes to reach this height.
(b) Determine the initial velocity with which the object sprang from the ground.
(c) If the object sprang from the ground with the same initial velocity but under the influence of a different gravity, then how high would it go and how long would it take to reach the top?

Gravity

The acceleration due to gravity on Earth is about 32.174 ft/s² (or about 9.8 m/s²). For the sake of these problems, we will use \( g \approx 32 \text{ ft/s}^2 \).

For aerodynamic objects falling short distances, there will not be much air resistance; so we can assume that the only force acting on the object after initial launch is \( g \). However, for longer distances or for a non-aerodynamic object, there will be considerable air drag that slows the object down and causes a terminal speed, which is the maximum (downward) speed that the object can attain. Below are some approximate terminal speeds.

<table>
<thead>
<tr>
<th>Body</th>
<th>Terminal Speed (ft/s)</th>
<th>Terminal Speed (mph)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Skydiver (Vertical)</td>
<td>278.8</td>
<td>190.1</td>
</tr>
<tr>
<td>Skydiver (spread eagle)</td>
<td>180.4</td>
<td>123</td>
</tr>
<tr>
<td>Parachutist</td>
<td>21.32</td>
<td>14.54</td>
</tr>
<tr>
<td>Ping Pong Ball</td>
<td>22.96</td>
<td>15.65</td>
</tr>
<tr>
<td>Baseball</td>
<td>131.2</td>
<td>89.45</td>
</tr>
<tr>
<td>Golfball</td>
<td>98.4</td>
<td>67.1</td>
</tr>
<tr>
<td>Stone (1 cm rad)</td>
<td>98.4</td>
<td>67.1</td>
</tr>
<tr>
<td>Raindrop</td>
<td>32.8</td>
<td>22.36</td>
</tr>
</tbody>
</table>

Note: On the Moon, there is no atmosphere; hence, there will be no air resistance on falling objects. The gravity on the moon is about 5.4 ft/s².
The Functions

We let \( h_0 \) denote the initial height (in feet) and let \( v_0 \) denote the initial velocity (in ft/s). The variable will be time \( t \) measured in seconds. Using \( g \approx 32 \text{ ft/s}^2 \), the height, velocity, and speed functions, for \( t \geq 0 \), are:

<table>
<thead>
<tr>
<th>Height (in feet)</th>
<th>Velocity (in feet per sec)</th>
<th>Speed (in feet per sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h(t) = -\frac{g}{2}t^2 + v_0 t + h_0 )</td>
<td>( v(t) = -gt + v_0 )</td>
<td>( s(t) =</td>
</tr>
<tr>
<td>( = -16t^2 + v_0 t + h_0 )</td>
<td>( = -32t + v_0 )</td>
<td>( =</td>
</tr>
</tbody>
</table>

\( v_0 > 0 \) for an object launched upward
\( v_0 = 0 \) for a dropped object
Max height occurs when \( v(t) = 0 \)
Object hits ground when \( h(t) = 0 \)

Velocity is positive for an object going upward and negative for an object going downward. But the speed is always positive; so speed is just the absolute value of velocity.

**Example 1.** Drop an (aerodynamic) object from 30 feet. (a) Solve for the time \( T \) it takes to fall 20 feet. How fast is it going at this time? (b) How long does it take to hit the ground, and what is its impact speed in mph?

**Solution.** When dropping an object, we let \( v_0 = 0 \). Using \( h_0 = 30 \text{ ft} \), we have

\[
\begin{align*}
  h(t) &= -16t^2 + 30 \\
  v(t) &= -32t \\
  s(t) &= 32t 
\end{align*}
\]

(a) At time \( T \), the object has fallen 20 feet, so the height is \( 30 - 20 = 10 \text{ ft} \). So now we set \( h(T) = 10 \) and solve for the time \( T \). Then we evaluate the speed at this time by multiplying by 32 ft/s².

\[
\begin{align*}
  h(T) &= 10 \\
  -16T^2 + 30 &= 10 \\
  -16T^2 &= -20 \\
  T^2 &= \frac{20}{16} \\
  T &= \frac{\sqrt{20}}{4} \approx 1.118 \text{ sec}
\end{align*}
\]

The speed at this time is \( 32 \times \frac{\sqrt{20}}{4} = 8\sqrt{20} \approx 35.777 \text{ ft/s} \).

(b) Now we repeat the work done in (a), but we first set the height equal to 0 to indicate that the object has hit bottom. To convert ft/s to mph, multiply by 3600/5280.

\[
\begin{align*}
  h(t) &= 0 \\
  -16t^2 + 30 &= 0 \\
  -16t^2 &= -30 \\
  t^2 &= \frac{30}{16} \\
  t &= \frac{\sqrt{30}}{4} \approx 1.3693 \text{ sec}
\end{align*}
\]

The speed at this time is \( 32 \times \frac{\sqrt{30}}{4} = 8\sqrt{30} \approx 43.8178 \text{ ft/s} \times \frac{3600}{5280} \approx 29.87 \text{ mph} \).

**Note:** The time it takes to fall \( K \) feet is always given by \( \sqrt{K} / 4 \) sec.
Example 2. An object is launched straight upward from 80 feet at 56 ft/s.

(a) With no air resistance, what are the height, velocity, and speed functions?
(b) What is the maximum height of the object?
(c) At what times does the object’s speed equal 40 ft/sec?
(d) In total, how long does it take for the object to hit the ground?
(e) What is the object’s speed when it hits the ground?

Solution. (a) Using \( v_0 = 56 \) and \( h_0 = 80 \), we have

\[
  h(t) = -16t^2 + 56t + 80 \quad v(t) = -32t + 56 \quad s(t) = |-32t + 56 |
\]

(b) We first must find the time it takes to reach the maximum height, which occurs when the velocity equals 0. So we solve \( v(t) = 0 \), and then substitute the time obtained into the height function \( h(t) \).

\[
  v(t) = 0 \quad \rightarrow \quad -32t + 56 = 0 \quad \rightarrow \quad -32t = -56 \quad \rightarrow \quad t = \frac{56}{32} = 1.75 \text{ sec}
\]

which gives the time of reaching the maximum height. Then the max height is \( h(1.75) = -16(1.75)^2 + 56(1.75) + 80 = 129 \text{ ft} \).

(c) There are two times when the speed equals 40 ft/s. Once on the way up when it has slowed from 56 ft/s down to 40 ft/s, and again on the way down when the velocity reaches −40 ft/s. To find these times, we must solve the equation

\[
  s(t) = |-32t + 56| = 40 \quad \text{or} \quad v(t) = -32t + 56 = \pm 40
\]

\[
  \rightarrow -32t = \pm 40 - 56 \quad \rightarrow \quad t = \frac{40 - 56}{-32} = 0.5 \quad \text{or} \quad t = \frac{-40 - 56}{-32} = 3
\]

Thus, at times 0.5 sec and 3 sec, the speed is 40 ft/s.

(d) To find the time needed to hit the ground, we set \( h(t) = 0 \) and solve for \( t \) with the quadratic formula or by factoring if possible. In any case, take only the positive solution for \( t \):

\[
  -16t^2 + 56t + 80 = 0 \quad \rightarrow \quad -8(2t^2 - 7t - 10) = 0 \quad \rightarrow \quad 2t^2 - 7t - 10 = 0
\]

\[
  t = \frac{-( -7) \pm \sqrt{(-7)^2 - 4(2)(-10)}}{2(2)} = \frac{7 \pm \sqrt{129}}{4} \quad \rightarrow \quad \frac{7 + \sqrt{129}}{4} \approx 4.5895 \text{ sec}
\]

Easier: The total time to hit is \( 1.75 + \frac{\sqrt{129}}{4} \) (time up + time to fall from max height)

(e) The impact velocity is \( -32 \times 4.5895 + 56 \approx -90.864 \text{ ft/s} \), which gives an impact speed of 90.864 ft/s. Or simply use \( 32 \times \sqrt{129} / 4 \approx 90.86 \text{ ft/s} \).
Example 3. A ball bounces straight upward to a maximum height of 8 feet.

(a) Solve for the time $T$ it takes to reach the maximum height and for the initial velocity with which it sprang from the floor.

(b) With the same initial velocity found in Part (a), how high would the ball bounce on the Krypton, where gravity is about 60 ft/s², and how long would it take to get there?

(c) What would happen on Tatooine where gravity is only 12 ft/s²?

Solution. For an object springing from the ground, we use $h_0 = 0$. And the initial velocity $v_0$ is unknown, so the height and velocity functions are

$$h(t) = -16t^2 + v_0 t \quad \text{and} \quad v(t) = -32t + v_0.$$ 

The maximum height of 8 feet occurs when $v(T) = 0$ at time $T$. So we can set $h(T) = 8$ and the $v(T) = 0$ to solve for the time $T$ and initial velocity $v_0$.

$$-16 T^2 + v_0 T = 8$$
$$-32 T + v_0 = 0 \quad \rightarrow \quad v_0 = 32 T \quad \rightarrow \quad -16 T^2 + (32 T) T = 8$$
$$\rightarrow -16 T^2 + 32 T^2 = 8 \quad \rightarrow \quad 16 T^2 = 8$$
$$\rightarrow T^2 = \frac{8}{16} = 0.5 \quad \rightarrow \quad T = \sqrt{0.5} \approx 0.7071 \text{ sec}$$

Then, $v_0 = 32 T = 32\sqrt{0.5} \approx 22.6274 \text{ ft/s}$ is the initial velocity with which the ball left the floor.

(b) Using the time of $\sqrt{0.5}$ sec and the height of 8 ft of the bouncing ball on Earth, we simply multiply by the ratio of gravities $32/60$ to get the time and height of a ball having the same initial velocity on Krypton:

$$t = \sqrt{0.5} \times \frac{32}{60} = 0.377 \text{ sec} \quad \text{and} \quad h = 8 \times \frac{32}{60} \approx 4.267 \text{ ft}.$$ 

So the stronger gravity greatly reduces the time and height.

(c) With a weaker gravity of 12 ft/s² on Tatooine, the time and height increase:

$$t = \sqrt{0.5} \times \frac{32}{12} = 1.8856 \text{ sec} \quad \text{and} \quad h = 8 \times \frac{32}{12} \approx 21.33 \text{ ft}.$$
Exercises

1. If you bungee jump from 100 ft, how fast (in mph) will you be going when you have fallen 75 ft? If you (unfortunately) hit the ground, at what speed will you splat?

2. An object is dropped and it takes 2.45 seconds to hit the ground. Estimate the original height of the object and the impact speed.

3. An object is thrown straight upward (on Earth) from 192 ft at 104 ft/sec.
   (a) With no air resistance, what are the height, velocity, and speed functions?
   (b) What is the maximum height of the object?
   (c) At what times does the object’s speed equal 60 ft/sec?
   (d) In total, how long does it take for the object to hit the ground?
   (e) What is the object’s speed in mph when it hits the ground?

4. A ball bounces straight upward to a maximum height of 4.5 feet.
   (a) Solve for the time $T$ it takes to reach the maximum height and for the initial velocity with which it sprang from the floor.
   (b) With the same initial velocity found in Part (a), how high would the ball bounce on the Moon where gravity is about 5.4 ft/s$^2$ and how long would it take to get there?
Solutions

1. Assume $v_0 = 0$ so that you are simply dropped from $h_0 = 100$ ft. Then

$$h(t) = -16t^2 + 100 \quad \text{and} \quad v(t) = -32t$$

When you have fallen 75 ft, then your height is 25 ft. We need to solve for this time $T$ when $h(T) = 25$, and then evaluate the speed at this time.

$$h(T) = 25 \quad \rightarrow \quad -16T^2 + 100 = 25 \quad \rightarrow \quad -16T^2 = -75 \quad \rightarrow \quad T = \frac{\sqrt{75}}{4} = 2.165 \text{ sec}.$$ 

Your speed at this time is $32 \times \frac{75}{4} \approx 69.28 \text{ ft/s}$, or $(\times \frac{3600}{5280}) \approx 47.24 \text{ mph}$.

If you hit the ground, then $h(t) = 0$. We now solve for this time, then evaluate the impact speed at this time.

$$h(t) = 0 \quad \rightarrow \quad -16t^2 + 100 = 0 \quad \rightarrow \quad -16t^2 = -100 \quad \rightarrow \quad t = \frac{\sqrt{100}}{4} = 2.5 \text{ sec}.$$ 

Your speed at 2.5 sec is about $32 \times 2.5 = 80 \text{ ft/s}$, or $(\times \frac{3600}{5280}) \approx 54.54 \text{ mph}$.

2. Because the object is dropped, we use $v_0 = 0$. Thus, $h(t) = -16t^2 + h_0$ and $v(t) = -32t$.

The height becomes 0 when $t = 2.45$; so we substitute this time for $t$ while setting the height equal to 0:

$$-16(2.45)^2 + h_0 = 0 \quad \rightarrow \quad h_0 = 16(2.45)^2 \approx 96 \text{ ft}.$$ 

The impact velocity is $-32 \times 2.45 = -78.4 \text{ ft/s}$; so the impact speed is about $78.4 \text{ ft/s}$ or $53.45 \text{ mph}$.

3. (a) $h(t) = -16t^2 + 104t + 192 \quad \text{v}(t) = -32t + 104 \quad \text{s}(t) = \left| -32t + 104 \right|$ 

(b) Solve $v(t) = 0 \rightarrow -32t + 104 = 0 \rightarrow t = 3.25 \text{ sec}$, which gives the time of reaching the maximum height. So the maximum height is $h(3.25) = 361 \text{ ft}$.

(c) Solve $v(t) = \pm 60 \rightarrow -32t + 104 = \pm 60 \rightarrow -32t = -44$ or $-32t = -164$ 

Thus, at times $t = 44/32 = 1.375 \text{ sec}$ and $t = 164/32 = 5.125 \text{ sec}$, the speed is 60 ft/s.

(d) The total time to hit the ground is $t = 3.25 + \frac{\sqrt{361}}{4} = 8 \text{ sec}$.

(e) The impact speed upon falling from the max height is $32 \times \sqrt{361} / 4 = 152 \text{ ft/s}$ 

$\times (3600/5280) = 103.636 \text{ mph}$. 

For the bouncing ball, we use \( h_0 = 0 \). So now \( h(t) = -16t^2 + v_0 t \) and \( v(t) = -32t + v_0 \).

The initial velocity \( v_0 \) and the time \( T \) to reach 4.5 ft in height are unknown. But we can set the height equal to 4.5 and the velocity equal to 0 to solve for \( T \) and \( v_0 \).

\[
-16 T^2 + v_0 T = 4.5 \\
-32 T + v_0 = 0 \quad \rightarrow \quad v_0 = 32 T \
\]

sub into 1st eq

\[
-16 T^2 + (32 T) T = 4.5 \quad \rightarrow \quad 16 T^2 = 4.5
\]

\[
\rightarrow T^2 = \frac{4.5}{16} \quad \rightarrow \quad T = \frac{\sqrt{4.5}}{4} \approx 0.53 \text{ sec}
\]

Then, \( v_0 = 32 T = 32 \times \frac{\sqrt{4.5}}{4} = 8\sqrt{4.5} \approx 16.97 \text{ ft/s} \) is the initial velocity with which the ball left the floor.

(b) Using the time of \( T = \frac{\sqrt{4.5}}{4} \) sec and the height of 4.5 ft of the bouncing ball on Earth, we simply multiply by the ratio of gravities 32/5.4 to get the time and height of a ball having the same initial velocity on the Moon:

\[
t = \frac{\sqrt{4.5}}{4} \times \frac{32}{5.4} \approx 3.1427 \text{ sec} \quad \text{and} \quad h = 4.5 \times \frac{32}{5.4} \approx 26.67 \text{ ft}.
\]

(Note: Because the Moon has a weaker gravity, the time should be longer and the height greater, which is why we multiply by 32/5.4 rather than 5.4/32.)