An annuity is a popular and effective method of saving for retirement. In a sense, it is the opposite of a mortgage. First, an initial deposit of $P$ is made. At the end of the month and at the end of each month thereafter, a monthly deposit of $M$ is made. Each month, you increase the principal upon which you are drawing interest, unlike a mortgage on which each month you decrease the principal upon which you are paying interest.

Assuming a final deposit is made at the end of the last month, the future value $FV$ in $t$ years is given by the formula:

$$ FV = P \left(1 + \frac{r}{12}\right)^{(12 \times t)} + \frac{12M}{r} \left[ \left(1 + \frac{r}{12}\right)^{(12 \times t)} - 1 \right] $$

**Note:** The future value is the annuity's worth after $t$ years. The present value is the amount that should be deposited now to give the same future value without having to make monthly payments.

**Example 1.** Suppose you start an annuity by depositing $30,000 and adding $500 a month for the next 30 years. The annuity pays 6.6% interest.

(a) What is the future value in 30 years and how much is actually paid in?
(b) What is the present value of this annuity?
(c) If you wanted $1,000,000 after 30 years but still only added $500 a month, then what should the initial deposit have been?
(d) If you wanted $1,000,000 after these 30 years and still deposited only $30,000, then how much should the monthly payment have been?

**Solution.** (a) The future value in 30 years is given by

$$ FV = 30000 \times \left(1 + \frac{0.066}{12}\right)^{(12 \times 30)} + 12 \times 500 \times \frac{0.066}{12} \left[ \left(1 + \frac{0.066}{12}\right)^{(12 \times 30)} - 1 \right] $$

$$ = 30000 \times (1.0055)^{360} + \frac{6000}{0.066} \left[ (1.0055)^{360} - 1 \right] $$

$$ = 216,106.70 + 563,959.70 $$

$$ = $780,066.40. $$

The amount paid in is $30,000 + 30 \times 12 \times 500 = $210,000. So you earn over $570,000 in interest over the course of 30 years.

(b) The present value is the required deposit to attain $780,066.40 in 30 years with no monthly payments. So we solve for $P$ in the equation

$$ P \times (1.0055)^{360} = 780,066.40, $$

which gives

$$ P = 780,066.40 \div (1.0055)^{360} = $108,289.07. $$
(c) We now solve for \( P \) in the equation: \( P \times (1.0055)^{360} + 563,959.70 = 1,000,000 \)
\[
\rightarrow P \times (1.0055)^{360} = 436,040.30 \quad \rightarrow \quad P = 436,040.30 \times (1.0055)^{360} = \$60,531.25.
\]

(d) Solve for \( M \) in the equation \( 216,106.70 + \frac{12 \times M}{0.066} (1.0055)^{360} - 1 = 1,000,000 \)
\[
\rightarrow \frac{12 \times M}{0.066} (1.0055)^{360} - 1 = 783893.30 \quad \rightarrow \quad M = 783893.30 \times 0.066 + 12 \times [(1.0055)^{360} - 1] \quad \rightarrow \quad M = \$694.99.
\]

**Simple Interest Annuities**

For annuities that pay simple interest, the additional payments are added just at the end of the year. In this case, the future value \( FV \) in \( t \) years is given by the formula:

\[
FV = P(1 + r)^t + \frac{M}{r}[(1 + r)^t - 1]
\]

**Example 2.** You begin a Roth IRA with a deposit of $3000. At the end of every year thereafter, you add another $3000. The IRA guarantees a minimum 4.5% return each year. What is the minimum future value in 25 years? How much is paid in over 25 years? What is the present value?

**Solution.** The future value is given by
\[
FV = P(1 + r)^t + \frac{M}{r}[(1 + r)^t - 1] = 3000 \times (1.045)^{25} + \frac{3000}{0.045}[(1.045)^{25} - 1] = \$142,711.93.
\]

To attain this amount, you have only paid in \( $3000 + 25 \times 3000 = $78,000 \).

**Note:** The interest gained on such a Roth IRA is tax-free! However the contributions made each year are not tax-deductible.

To find the present value, solve for \( P \) in the equation: \( P \times (1.045)^{25} = 142,711.93 \quad \rightarrow \quad P = 142,711.93 + (1.045)^{25} = \$47,484.63 \). In other words, if you invest \$47,484.63 with simple yearly interest of 4.5%, then you will attain $142,711.93 in 25 years without any additional yearly payments.

**Exercise.** You start an annuity by depositing $5000 and adding $200 a month for the next 25 years. The annuity pays 6% interest.
(a) What is the future value in 25 years and how much is actually paid in?
(b) What is the present value of this annuity?
(c) If you wanted $200,000 after 25 years but still only added $200 a month, then what should the initial deposit have been?
(d) If you wanted $200,000 after these 25 years and still deposited only $5000, how much should the monthly payment have been?
Solution

(a) The future value in 25 years is

\[ FV = 5000 \times \left(1 + \frac{0.06}{12}\right)^{(12 \times 25)} + \frac{12 \times 200}{0.06} \left[\left(1 + \frac{0.06}{12}\right)^{(12 \times 25)} - 1\right] \]

\[ = 5000 \times (1.005)^{300} + 40,000 \times [(1.005)^{300} - 1] \]

\[ = 22,324.85 + 138,598.79 \]

\[ = \$160,923.64. \]

The amount paid in is \( 5000 + 25 \times 12 \times 200 = \$65,000. \)

(b) The present value is the initial amount \( P \) that could be deposited with no monthly payments in order to achieve the same future value.

\[ P \times (1.005)^{300} = 160,923.64 \quad \rightarrow \quad P = 160,923.64 + (1.005)^{300} = \$36,041.37. \]

(c) Solve for \( P \) in the equation \( P \times (1.005)^{300} + 138,598.79 = 200,000 \)

\[ \rightarrow \quad P \times (1.005)^{300} = 61,401.21 \]

\[ \rightarrow \quad P = 61,401.21 + (1.005)^{300} \quad \rightarrow \quad P = \$13,751.76. \]

(d) Solve for \( M \) in the equation \( 22,234.85 + \frac{12M}{0.06} \left[(1.005)^{300} - 1\right] = 200,000. \)

\[ \rightarrow \quad \frac{12M}{0.06} \left[(1.005)^{300} - 1\right] = 177,675.15 \]

\[ \rightarrow \quad M = 177,675.15 \times 0.06 + 12 + [(1.005)^{300} - 1] \quad \rightarrow \quad M = \$256.39. \]