Measurements that are normally distributed can be described in terms of their mean $\mu$ and standard deviation $\sigma$. These measurements should have the following properties:

(i) The mean and mode both equal the median; that is, the average value and the most likely value are both in the middle of the distribution.

(ii) The measurements are symmetric about the mean.

(iii) (The 68–95–99.7 Rule): Around 68% of the measurements should be within one standard deviation of average, around 95% should be within two standard deviations of average, and around 99.7% of the measurements should be within three standard deviations of average.

(iv) A histogram of measurements create a “Bell-Shaped Curve” with the percentages at the high and low ends dropping off exponentially.

Such a measurement is denoted by $X \sim N(\mu, \sigma)$. When $\mu = 0$ and $\sigma = 1$, then we have the standard normal distribution that is denoted by $Z \sim N(0, 1)$.

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**Example 1.** In the U.S., birth weights are normally distributed with a mean of 7 pounds and a standard deviation of 2 pounds. Explain what this means in terms of the properties of a normal distribution.

**Solution.** Let $\Omega$ be all babies born in the U.S., and let $X$ denote the birth weight. Then $X \sim N(7, 2)$ lbs. That is,

(i) In the U.S, the average birth weight, the most likely birth weight, and the median birth weight are all 7 lbs.

(ii) Birth weights as a whole are symmetric about the weight 7 lbs.

(iii) Around 68% of newborn weights are from 5 to 9 lbs ($\mu \pm \sigma$); around 95% of newborn weights are from 3 to 11 lbs ($\mu \pm 2\sigma$); and around 99.7% of newborn weights are from 1 to 13 lbs ($\mu \pm 3\sigma$).

(iv) A histogram of newborn birth weights create a “Bell-Shaped Curve” with the percentages of high and low weights dropping off exponentially.
**Example 2** A hospital weighs all the babies that are born in the maternity ward. The weights in pounds for one particular week are as follows:

<table>
<thead>
<tr>
<th>5.64</th>
<th>5.39</th>
<th>5.86</th>
<th>7.99</th>
<th>7.89</th>
<th>11.27</th>
<th>7.31</th>
<th>6.54</th>
<th>7.53</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.15</td>
<td>2.93</td>
<td>7.08</td>
<td>5.08</td>
<td>7.46</td>
<td>9.71</td>
<td>9.17</td>
<td>7.50</td>
<td>7.10</td>
</tr>
<tr>
<td>9.30</td>
<td>4.69</td>
<td>8.38</td>
<td>2.74</td>
<td>6.02</td>
<td>6.94</td>
<td>6.89</td>
<td>6.33</td>
<td>6.07</td>
</tr>
<tr>
<td>5.96</td>
<td>8.05</td>
<td>6.81</td>
<td>8.04</td>
<td>6.57</td>
<td>7.26</td>
<td>6.88</td>
<td>8.05</td>
<td>4.91</td>
</tr>
</tbody>
</table>

Do these weights actually appear to be normally distributed?

Here let $\Omega$ be all babies born at *this* hospital and let $X$ be the birth weight. To study the sample, we first shall construct a histogram with range [2, 12] on the $x$-axis with bins of length 0.5.

We see that the histogram somewhat resembles a bell-shaped curve, with symmetry about the middle, the most likely values in the middle, and the numbers of measurements somewhat tailing off at both extreme ends.

If we compute the statistics, we obtain a mean of $\mu \approx 6.819$, a standard deviation of $\sigma \approx 1.733$, and a median of 6.915. So the mean is close to the median, and the most likely values occur in the bin from 6.5 lbs. to 7 lbs. which also contains the mean and the median. Thus, it may be safe to say that the mean and mode both equal the median.

The range $\mu \pm \sigma$ is 5.086 to 8.552 lbs. and contains 26 out of 36 or about 72.2% of the weights.

The range $\mu \pm 2\sigma$ is 3.353 to 10.285 lbs. and contains 33 out of 36 or about 91.67% of the weights. The range $\mu \pm 3\sigma$ is 1.62 to 12.018 lbs. and contains 100% of the weights.

So the 68-95-99.7 rule is slightly violated, but overall these baby weights seem to be *close* to normally distributed. By including *many more* measurements over more weeks, we would possibly come to a stronger belief that new born baby weights are in fact normally distributed.
Normal Calculations

Given a normal distribution \( X \sim N(\mu, \sigma) \), we wish to find various probabilities of where an arbitrary measurement may lie. For instance, we could find \( P(a \leq X \leq b) \), which is the probability that a random measurement \( X \) lies between \( a \) and \( b \).

\[
P(a \leq X \leq b) = \text{normalcdf}(a, b, \mu, \sigma)
\]

\[
P(X \leq k) = \text{normalcdf}(\mu, k, \mu, \sigma)
\]

\[
P(X > k) = \text{normalcdf}(k, \infty, \mu, \sigma)
\]

The bounds \(-1 \times 10^9\) and \(1 \times 10^9\) are used as “estimates” of \(-\infty\) and \(+\infty\).

Inverse Normal Calculation

To find the value \( x \) for which \( P(X \leq x) \) equals a desired proportion \( p \) (an inverse normal calculation), we use the command \text{invNorm}(p, \mu, \sigma). The \text{invNorm} command is also found in the DISTR (2nd Vars) menu.

Example 3. The lengths of human pregnancies are approximately normally distributed with a mean of 266 days and a standard deviation of 16 days.

(a) What is the population \( \Omega \)? What is the measurement \( X \) and its distribution?
(b) What percent of pregnancies last at most 240 days?
(c) What percent of pregnancies last from 240 to 270 days?
(d) How long do the longest 20% of pregnancies last?

Solution. (a) Here \( \Omega \) is the population of all women who have given birth and \( X \) is the measurement of how many days the pregnancy lasted. Then \( X \approx N(266, 16) \).
(b) For $X \sim N(266, 16)$, we wish to find $P(X \leq 240)$. We use the command `normalcdf(-1E99, 240, 266, 16)`. We see that around 5.2% of pregnancies last at most 240 days.

(c) To find $P(240 \leq X \leq 270)$, enter the command `normalcdf(240, 270, 266, 16)`. We see that around 54.66% of pregnancies last from 240 to 270 days.

(d) To find how long the longest 20% of pregnancies last, we must find the value $x$ for which $P(X \geq x) = 0.20$. But to use the `invNorm` command, we instead must find $x$ such that $P(X \leq x) = 0.80$. Using the command `invNorm(.80, 266, 16)`, we see that the longest 20% of pregnancies last at least 279.46 days.

Example 4. Heights of adult women are normally distributed with a mean of 65.5 inches and a standard deviation of 2.75 inches. What percentage of women are

(a) at least 70 in. tall?   (b) at most 63 in. tall?  (c) from 64 to 68 in. tall?

(d) What height is such that 95% of all women are below this height?

(e) What height is such that 90% of all women are above this height?

(f) What two heights, symmetric about the mean, contain 50% of all heights?

Solution. Here, $\Omega$ = All adult women and $X =$ height in inches. Then $X \sim N(65.5, 2.75)$.

(a) At least 70, meaning 70 or more: Enter the command `normalcdf(70, 1E99, 65.5, 2.75)` to obtain $P(X \geq 70) \approx 0.05088$. So about 5.09% of women are at least 70 inches tall.

(b) At most 63, meaning up to 63: Enter the command `normalcdf(-1E99, 63, 65.5, 2.75)` to obtain $P(X \leq 63) \approx 0.18165$. So about 18.165% of women are at most 63 inches tall.

(c) $P(64 \leq X \leq 68) \approx 0.5256$. So about 52.56% of women are from 64 to 68 inches tall.
(d) We must find \( x \) such that \( P(X < x) = 0.95 \). To do so, enter `invNorm(.95, 65.5, 2.75)` to obtain \( x \approx 70 \) inches.

(e) We must find \( x \) such that \( P(X > x) = 0.90 \) or equivalently such that \( P(X \leq x) = 0.10 \). To do so, enter `invNorm(.10, 65.5, 2.75)` to obtain \( x \approx 61.976 \) inches.

(f) If we want 50% of heights in the middle, then we need \( x \) and \( y \) such that \( P(x \leq X \leq y) = 0.50 \). But then we need 25% of the heights at each tail. So we need \( x \) such that \( P(X \leq x) = 0.25 \) and \( y \) such that \( P(X \leq y) = 0.75 \). Enter `invNorm(.25, 65.5, 2.75)` to obtain \( x \approx 63.645 \) inches and `invNorm(.75, 65.5, 2.75)` to obtain \( y \approx 67.355 \) inches.

**Standard Normal Distribution**

Suppose \( X \sim N(\mu, \sigma) \) is a normally distributed measurement. For example IQ scores are such that \( X \sim N(100, 15) \). Then most measurements (about 99.7%) are within \( \mu \pm 3\sigma \), which is 55 to 145 for IQ scores. If we subtract \( \mu \) from every measurement, then we still have a normal distribution, but most values will be between \(-3\sigma\) and \(3\sigma\). By subtracting \( \mu \), the result is \( N(0, \sigma) \).

Next, suppose we divide the new values by \( \sigma \). Then most values will be between \(-1\) and \(1\). The result is now \( N(0, 1) \). By subtracting \( \mu \) from every measurement and then dividing by \( \sigma \), we have *standardized* the values and have obtained the standard normal distribution \( Z \sim N(0, 1) \).

Let \( X \sim N(\mu, \sigma) \) and \( Z = \frac{X - \mu}{\sigma} \).

Then \( Z \sim N(0, 1) \).

For \( Z \sim N(0, 1) \), we still can compute the various probabilities and inverse calculations using the `normalcdf` and `invNorm` commands. For example, \( P(Z \geq -1.22) \) and \( P(-0.85 \leq Z \leq 1.05) \) are shown below.
**Example 5.** Let $Z \sim N(0, 1)$.

(a) Find the number $z$ such that $P(Z \geq z) = 0.05$.
(b) Find the numbers $w$ and $z$ such that $P(w \leq Z \leq z) = 0.95$.

**Solution.** For Part (a), we actually need $P(Z \leq z) = 0.95$. So enter the command `invNorm(.95, 0, 1)` to obtain $z \approx 1.645$.

For Part (b), we need $w$ and $z$ such that $P(Z \leq w) = 0.025$ and $P(Z \leq z) = 0.975$. So enter the commands `invNorm(.025, 0, 1)` and `invNorm(.975, 0,1)` to obtain $w \approx -1.96$ and $z \approx 1.96$.

By converting different normal distributions $X$ and $Y$ to standard normal distributions, then $X$ and $Y$ can be placed on the same scale. Values from $X$ and $Y$ then can be compared without any probability calculation.

**Example 6.** IQ scores are $X \sim N(100, 15)$ and baby birth weights are $Y \sim N(7, 2)$ (lbs). Which is less likely, an IQ of at least 145 or a new-born weighing at least 10 pounds?

**Solution.** Simply convert each value to a standard normal scale:

\[
X \geq 145 \rightarrow Z \geq \frac{145 - 100}{15} \rightarrow Z \geq 3 \quad Y \geq 10 \rightarrow Z \geq \frac{10 - 7}{2} \rightarrow Z \geq 1.5
\]

The range $Z \geq 3$ creates less probability than $Z \geq 1.5$, so an IQ of at least 145 is less likely than a newborn weighing at least 10 pounds.
Practice Exercises

1. Students in a Psychology Masters Program are given an IQ test. The scores are generally found to be normally distributed with a mean of 112 and a standard deviation of 9.

(a) Give the population Ω under consideration and the measurement X. What is the notation for the distribution of X?

(b) Compute the probability that a random score is

(i) At most 105       (ii) From 100 to 124       (iii) At least 130.

(c) What scores x and y are such that

(i) Only 25% score below x       (ii) 10% score above y?

(d) What scores x and y, symmetric about the mean, are such that \( P(x \leq X \leq y) = 0.66 \)?

2. Let \( Z \sim N(0, 1) \).

(a) Compute

(i) \( P(Z \geq 1.45) \)       (ii) \( P(Z \leq -2.12) \)       (iii) \( P(-2 \leq Z \leq 2) \)

(b) Find the numbers \( w \) and \( z \) such that \( P(w \leq Z \leq z) = 0.97 \).

(c) Find the number \( z \) such that \( P(Z \geq z) = 0.01 \).

3. ACT scores are \( X \sim N(22.4, 3.2) \) and SAT scores are \( Y \sim N(1020, 160) \). (a) Which is a better score, an ACT of 28 or an SAT of 1400? (b) Which happens less often, an ACT of at most 14 or an SAT of at least 1400?
Answers

1. \( \Omega = \text{All students in this Psychology Masters Program}; \ X = \text{IQ score}; \ X \sim N(112, 9) \)

   (b) \( P(X \leq 105) \approx 0.21835 \quad P(100 \leq X \leq 124) \approx 0.817577 \quad P(X \geq 130) \approx 0.02275 \)

   (c) \( x = \text{invNorm(.25, 112, 9)} \approx 105.93 \quad \text{and} \quad y = \text{invNorm(.90, 112, 9)} \approx 123.534 \)

   (d) \( x = \text{invNorm(.17, 112, 9)} \approx 103.4125 \quad \text{and} \quad y = \text{invNorm(.83, 112, 9)} \approx 120.5875 \)

2. (a) \( P(Z \geq 1.45) \approx 0.07353 \quad P(Z \leq -2.12) \approx 0.017 \quad P(-2 \leq Z \leq 2) \approx 0.9545 \)

   (b) \( w \approx -2.17 \quad \text{and} \quad z \approx 2.17 \quad (\text{invNorm(.015, 0, 1)} \quad \text{and} \quad \text{invNorm(.985, 0, 1)}) \)

   (c) \( z = \text{invNorm(.99, 0, 1)} \approx 2.326 \)

3. (a) Convert each score to standard normal:

   \[ X = 28 \rightarrow Z = \frac{28 - 22.4}{3.2} \rightarrow Z = 1.75 \]

   \[ Y = 1400 \rightarrow Z = \frac{1400 - 1020}{160} \rightarrow Z = 2.375 \]

   So an SAT of \( Y = 1400 \) produces the higher score on a standard scale.

   (b) \( X \leq 14 \rightarrow Z \leq \frac{14 - 22.4}{3.2} \rightarrow Z \leq -2.625 \quad Y \geq 1400 \rightarrow Z \geq \frac{1400 - 1020}{160} \rightarrow Z \geq 2.375 \)

   Here, the left tail \( Z \leq -2.625 \) creates less probability than the right tail \( Z \geq 2.375 \); so an ACT score of at \( X \leq 14 \) is less likely to occur.