A *mathematical set* is a well-defined collection of objects $A$ for which we can determine precisely whether or not any object belongs to $A$. Objects in a set are formally called *elements* of the set. The elements that do not belong to $A$ are said to be in “A complement”, denoted by $A'$. It is always the case that $(A')' = A$.

When working with several sets, we denote them by different capital letters $A$, $B$, $C$, etc. For any particular problem, there is usually a *universal set* that narrows down the collection of all objects under consideration for that problem. The universal set is usually denoted by $U$ or $\Omega$. Then other sets under consideration are *subsets* of the universal set. A special subset is the empty set $\emptyset$ which contains nothing.

**Example 1.** Let $\Omega =$ All currently enrolled WKU students. Then $\Omega$ is a set because we can determine precisely whether or not any person is currently enrolled as a student at WKU. Now let $G =$ “Good” students and let $GS =$ Students who have at least a 2.00 cumulative GPA. Explain whether or not $G$ and $GS$ are sets.

The descriptor “good” is *not* well-defined because it is subjective, so $G$ is not a properly defined set. However for any student in $\Omega$, we can determine whether or not that student has at least a 2.00 GPA. So $GS$ is well-defined, and these students generally are classified as being in “good standing.”

**Example 2.** Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{4, 6, 7, 9, 10\}$, and $B = \{2, 6, 8, 9\}$. Draw the sets on a Venn diagram.

A Venn diagram of a universal set $U$ with two subsets $A$ and $B$ shows the elements of set $U$ partitioned into four regions. If the subsets $A$ and $B$ overlap, then we first determine which elements are in both subsets, and then write these elements within the overlap of two circles.

In this case, 6 and 9 are in both subsets. Then 4, 7, and 10 are in $A$ but not $B$; the elements 2 and 8 are in $B$ but not $A$; and the elements 1, 3, 5 are in neither $A$ nor $B$ but are still within the universal set.
**Example 3.** Let $\Omega$ be the 26 letters of the English alphabet (non-case sensitive). Let $F$ be the letters in the name “Ferdinand” and let $I$ be the letters in the name “Isabella.”

(a) Write the elements of $F$ and $I$ and show them on a Venn diagram.
(b) Make a block diagram that shows the number of elements in each region of the Venn diagram.

**Solution.** (a) When listing elements in a set, we do not list elements twice. So the elements of $F$ and $I$ are $F = \{f, e, r, d, i, n, a\}$ and $I = \{i, s, a, b, e, l\}$. The letters in common are $\{e, i, a\}$. The elements in $\Omega$ that are in neither $F$ nor $I$ are $\{c, g, h, j, k, m, o, p, q, t, u, v, w, x, y, z\}$. Thus a Venn diagram for these sets is:

(b) Set $F$ has 7 elements divided into two regions: 3 that in are both $F$ and $I$ and 4 that are in $F$ but not $I$. These three values create the top numerical row of the block diagram. Set $I$ has 6 elements divided into two regions: 3 that in are both $I$ and $F$ and 3 that are in $I$ but not $F$. These three values create the left numerical column of the block diagram. The total of 26 letters in $\Omega$ is placed in the bottom right.

The remainder of the block diagram can be obtained by subtracting from the total:
Set Notation Synopsis

**Name** | **Notation** | **Regions**
--- | --- | ---
Universal Set | $\Omega$ or $U$ | 1 + 2 + 3 + 4
Set $A$ | $A$ | 1 + 2
Not in $A$ (A complement) | $A'$ | 3 + 4
Set $B$ | $B$ | 2 + 3
Not in $B$ (B complement) | $B'$ | 1 + 4
$A$ or $B$ (A union B) | $A \cup B$ | 1 + 2 + 3
$A$ and $B$ (A intersect B) | $A \cap B$ | 2
$A$ but not $B$ (only A) | $A - B = A \cap B'$ | 1
$B$ but not $A$ (only B) | $B - A = B \cap A'$ | 3
Neither $A$ nor $B$ | $(A \cup B)' = A' \cap B'$ | 4
Not both $A$ and $B$ | $(A \cap B)' = A' \cup B'$ | 1 + 3 + 4
$A$ or $B$ but not both (exclusive or) (exactly one) | $A \Delta B = (A - B) \cup (B - A) = (A \cup B) - (A \cap B)$ | 1 + 3
Both or neither | $A \Delta B' = A' \Delta B = (A \cap B) \cup (A' \cap B')$ | 2 + 4
Not just $A$ by itself | $(A - B)' = (A \cap B)' = A' \cup B$ | 2 + 3 + 4
Not just $B$ by itself | $(B - A)' = (B \cap A')' = A \cup B'$ | 1 + 2 + 4
Set Operation Synopsis

**Complement:** means “Not”.

\[ A' \text{ – Regions 3, 4 } \quad \text{“Not in } A \text{”} \quad \quad B' \text{ – Regions 1, 4 } \quad \text{“Not in } B \text{”} \]

**Intersection \( \cap \):** Means “And”; that is, “Both traits at the same time”

There are four possible intersections. Each represents one region of the Venn or Block diagram.

\[ A \cap B \text{ – Region 2 } \quad \text{“Both } A \text{ and } B \text{”} \quad \quad A \cap B' \text{ – Region 1 } \quad \text{“} A \text{ but not } B \text{”} \]
\[ A' \cap B \text{ – Region 3 } \quad \text{“} B \text{ but not } A \text{”} \quad \quad A' \cap B' \text{ – Region 4 } \quad \text{“Neither } A \text{ nor } B \text{”} \]

**Other common notation:** (i) \( A \cap B' = A - B \) (ii) \( A' \cap B = B - A \).

**Example 4.** Let \( \Omega \) be adults (age 18 and older) living in the U.S.

(a) Let \( F \) = Females and \( A \) = Those who Approve of President’s Performance. (Then \( F' \) = Males, \( A' \) = Those who Do Not Approve of President’s Performance)

\[ F \cap A = \text{Females who Approve;} \quad F \cap A' = \text{Females who Do Not Approve;} \]
\[ F' \cap A = \text{Males who Approve;} \quad F' \cap A' = \text{Males who Do Not Approve.} \]

(b) Let \( M \) = Married and \( C \) = Have Children. (Then \( M' \) = Not Married, \( C' \) = Do not Have Children). The four intersections are

\[ M \cap C = \text{Married with Children} \]
\[ M \cap C' = M - C = \text{Married but Don't Have Children;} \]
\[ M' \cap C = C - M = \text{Not Married but Have Children;} \]
\[ M' \cap C' = \text{Not Married and Do Not Have Children.} \]

**Union \( \cup \):** Means “Or”, meaning “one or the other or possibly both”

There are four possible unions. Each represents three sections of the Venn or Block diagram.

\[ A \cup B \text{ – Regions 1, 2, 3 } \quad \text{“In } A \text{ (1, 2) or in } B \text{ (3, remainder of } B \text{)”} \]
\[ A \cup B' \text{ – Region 1, 2, 4 } \quad \text{“In } A \text{ (1, 2) or not in } B \text{ (4)”} \]
\[ A' \cup B \text{ – Regions 3, 4, 2 } \quad \text{“Not in } A \text{ (3, 4) or is in } B \text{ (2)”} \]
\[ A' \cup B' \text{ – Region 3, 4, 1 } \quad \text{“Not In } A \text{ (3, 4) or not in } B \text{ (1)”} \]
Again let $\Omega$ be adults (age 18 and older) living in the U.S.

(c) Let $F = \text{Females}$, $T = \text{At Most 6 ft Tall}$ (then $F' = \text{Males}$ and $T' = \text{Over 6 ft Tall}$). The four possible unions are

$$F \cup T = \text{adults who are female or are at most 6 feet tall}$$
(this set includes all females along with those males who are at most 6 ft)

$$F \cup T' = \text{adults who are female or are over 6 feet tall}$$
(this set includes all females along with those males who are over 6 ft)

$$F' \cup T = \text{adults who are male or are at most 6 feet tall}$$
(this set includes all males along with those females who at most 6 ft)

$$F' \cup T' = \text{adults who are male or are over 6 feet tall}$$
(this set includes all males along with those females who over 6 ft)

(d) Let $M = \text{Married}$ and $C = \text{Have Children}$.

$$M \cup C = \text{Married or Have Children;}$$
(this set includes all adults who are married along with those non-married adults who have children)

$$M \cup C' = \text{Married or Do Not Have Children;}$$
(this set includes all adults who are married along with those non-married adults who don't have children)

$$M' \cup C = \text{Not Married or Have Children;}$$
(this set includes all adults who are not married along with those married adults who have children)

$$M' \cup C' = \text{Not Married or Do Not Have Children;}$$
(this set includes all adults who are not married along with those married adults who don't have children)

**Exclusive Or $\Delta$:** Means “One or the other but not both”

Again there are four possible combinations, but two pairs will be the same. Each will represent two regions in the Venn or Block diagram.

$$A \Delta B, A' \Delta B'$$, both are Regions 1, 3 which gives “$A$ or $B$, but not both”

$$A \Delta B', A' \Delta B$$ both are Regions 2, 4. These regions are the same as

$$(A \cap B) \cup (A' \cap B')$$, which means “Both or Neither”

(e) Let $D = \text{Have a Dog}$ and $C = \text{Have a Cat}$.

$$D \Delta C = \text{have a dog or have a cat, but don't have both}$$

$D' \Delta C'$ is often awkward to express in words; so it is best to remember that it is the same as $D \Delta C$ which is easier to express.

$$D \Delta C' = D' \Delta C = \text{have both a dog and a cat, or have neither.}$$
(f) Let \( C = \) Support Capital Punishment, \( T = \) Support Term-limits.  
(Assume here that the complement of “support” is “oppose.”)

\[
C \Delta T = \text{support one or the other, but don't support both}  \\
C' \Delta T' = \text{oppose one or the other, but don't oppose both}  \\
\text{Logically, } C \Delta T = C' \Delta T'.
\]

\[
C' \Delta T = C \Delta T' = \text{means you either support both or you oppose both.}
\]

**DeMorgan’s Laws**

The complement of an intersection (1 region) must be three regions which is therefore a union. Likewise the complement of a union must be an intersection. The rules below are called *DeMorgan’s Laws*.

**The complement of an intersection is the union of the complements.**

\[
(A \cap B)' = A' \cup B'; \quad (A' \cap B)' = A \cup B; \quad (A \cap B')' = A' \cup B; \quad (A' \cap B')' = A \cup B.
\]

**The complement of a union is the intersection of the complements.**

\[
(A \cup B)' = A' \cap B'; \quad (A' \cup B)' = A \cap B'; \quad (A \cup B')' = A' \cap B; \quad (A' \cup B')' = A \cap B.
\]

**Cardinality, Proportion, and Percentage**

A set is generally denoted by a capital letter such as \( A \). But we often want to know how many elements are in set \( A \). This number of elements is called the *cardinality* of the set and is denoted by \( |A| \).

If we also know \( |\Omega| \), the number of elements in the entire universal set, and if \( |\Omega| < \infty \), then we can determine the proportion of elements in \( A \), denoted and given by

\[
P(A) = \frac{|A|}{|\Omega|}.
\]

This proportion gives the probability of choosing from set \( A \) when choosing an element at random from \( \Omega \). We often state \( P(A) \) as a percentage, but technically \( P(A) \) is always a value between 0 and 1.

**Example 5.** Let \( \Omega \) be a standard deck of cards, with no Jokers, and let \( H \) be the Hearts. What are \( |\Omega| \), \( |H| \), and \( P(H) \)?

Here \( |\Omega| = 52 \), \( |H| = 13 \), and \( P(H) = \frac{13}{52} = \frac{1}{4} = 0.25 \). We also may say that \( P(H) = 25\% \).
Example 6. Among students at WKU, let $E =$ those taking English and $M =$ those taking Math. Suppose 30% are taking English and 20% are taking Math. Moreover, 62% are taking neither.

(a) Show the percentages on a Block diagram.

Put each set in words and find the percentage of students in each set: (For (g) and (h), first simplify with DeMorgan’s Laws).

\[
\begin{align*}
(b) & \quad E' \cap M \\
(c) & \quad E' \cup M' \\
d & \quad E \cup M' \\
(e) & \quad E \Delta M \\
f & \quad E \Delta M' \\
g & \quad (E' \cup M)' \\
h & \quad (E \cap M')'
\end{align*}
\]

Put in symbols and find the percentage of students:

(i) taking English but not Math
(j) not taking English or are taking Math
(k) taking both or not taking either
(l) taking Math or English

(m) taking Math and English

Solution. The initial information is shown in the partial block diagram on the left below. By subtracting from the total, we can obtain the remainder of the diagram.

<table>
<thead>
<tr>
<th></th>
<th>$M$</th>
<th>$M'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$</td>
<td></td>
<td>30%</td>
</tr>
<tr>
<td>$E'$</td>
<td>20%</td>
<td>100%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$M$</th>
<th>$M'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$</td>
<td>12%</td>
<td>18%</td>
</tr>
<tr>
<td>$E'$</td>
<td>8%</td>
<td>62%</td>
</tr>
</tbody>
</table>

(b) $E' \cap M =$ taking Math but not taking English; $P(E' \cap M) = 8\%$
(c) $E' \cup M' =$ either not taking English or not taking Math; (a union is 3 inner regions) 
$P(E' \cup M') = 8 + 62 + 18 = 88\%$
(d) $E \cup M' =$ taking English or not taking Math; $P(E \cup M') = 12 + 18 + 62 = 92\%$
(e) $E \Delta M =$ taking English or Math, but not both; $P(E \Delta M) = 18 + 8 = 26\%$
(f) $E \Delta M' =$ taking both English and Math, or taking neither; 
$P(E \Delta M') = 12 + 62 = 74\%$
(g) $(E' \cup M)' = E \cap M' =$ taking English but not Math; $P(E \cap M') = 18\%$
(h) $(E \cap M')' = E' \cup M =$ taking Math or not taking English; 
$P(E' \cup M) = 12 + 8 + 62 = 82\%$

(i) taking English but not Math $= E \cap M'$; $P(E \cap M') = 18\%$

(j) not taking English or are taking Math $= E' \cup M$; 
$P(E' \cup M) = 8 + 62 + 12 = 82\%$

(k) taking both or not taking either $= (E \cap M) \cup (E' \cap M') = E' \Delta M = E \Delta M'$; 
$P(E \Delta M') = 74\%$

(l) taking Math or English $= E \cup M$; $P(E \cup M) = 12 + 18 + 8 = 38\%$

(m) taking Math and English $= E \cap M$; $P(E \cap M) = 12\%$
Exercises

1. Out of 150 people surveyed, a total of 80 are married and 78 have a child. Moreover, 32 are married but have no children. Let $M = \text{married}$ and $C = \text{have children}$.

   (a) Show all cardinalities on a complete Block diagram.

   Put the following events in set notation and find the cardinality of each set:

   (b) has children and is not married
   (c) has no children or is married
   (d) is neither married nor has children
   (e) is married or has children but not both.

   Put the following sets in words and find the probability of each set:

   (f) $(M \cap C)'$  (g) $M \cap C'$  (h) $(M \cap C) \cup (M' \cap C')$

2. Suppose 22% of adults favor more tax cuts and 44% oppose the death penalty. Moreover, 40% oppose both more tax cuts and the death penalty.

   Let $T = \text{favor more tax cuts}$  $D = \text{favor death penalty}$
   $T' = \text{oppose more tax cuts}$  $D' = \text{oppose death penalty}$

   (a) Show the percentages on a complete Block Diagram.

   Assume an adult is chosen at random. Put the following events in set notation and find the probability that the person:

   (b) favors more tax cuts and opposes the death penalty
   (c) either favors both policies or opposes both policies
   (d) opposes more tax cuts or opposes the death penalty
   (e) opposes more tax cuts or opposes the death penalty but doesn’t oppose both.

   Put the following sets in words and find the percentage in each set:

   (f) $T \Delta D$  (g) $(T \cup D)'$  (h) $(T' \cap D)'$