Variation of Species.

On p. 181 of Wallace's "Darwinism," ed. 1889, this passage occurs:—"Let us suppose that a given species consists of 100,000 individuals of each sex, with only the usual amount of fluctuating external variability. Let a physiological variation arise, so that 10 per cent. of the whole number—10,000 individuals of each sex—while remaining fertile inter se become quite sterile with the remaining 90,000. This peculiarity is not correlated with any external differences of form or colour, or with inherent peculiarities of likes or dislikes leading to any choice as to the pairing of the two sets of individuals. We have now to inquire, What would be the result?"

I have here attempted to investigate this question algebraically.

A. We shall suppose, as Dr. Wallace does, that the number of males in the species is the same as the number of females. Each of these numbers we shall denote by unity. For convenience, we shall speak of the number of either sex as the number of the species. Let then

\[ x = \text{the number of the normal species}, \]
\[ y = \text{the number of the variant variety}, \]

and

\[ x = 100,000, \quad y = 100,000. \]

B. We shall now consider the case when the unions between the two varieties are not sterile, and the hybrids are also fertile inter se and with the parent varieties.

\[ \text{Let the relative, effective, fertility of the hybrids and mongrels, inter se and with the parent varieties, be denoted by the factor } k, \text{ which we shall assume to be always less than unity. Also, let the effective fertility of the normal species in the production of variants be denoted by the factor } \mu; \text{ and let } z \text{ denote the number of the variant variety in the generation } (x, y, z). \]

Then the equations which determine the stable and permanent condition, if there be one, are

\[ \begin{align*}
(1 - \mu) x^2 & = y^2 + \mu z^2 = [x + y + z]^2 - x^2 - y^2 \cdot k \\
\frac{1}{x^2} & + \frac{1}{y^2} + \frac{1}{z^2} \cdot k \\
\end{align*} \]

Put \( a = 1 - z, \quad k' = 1 - k, \quad \mu' = 1 - \mu; \)

then

\[ \begin{align*}
\mu' (1 - \mu') = k, \\
\mu' x^2 = x^2 + (a - x)^2. \\
\end{align*} \]

Put \( \beta = (1 - \mu') / \mu; \)

then

\[ \beta^2 - k (3 - \mu) \beta + k^2 = 0. \]

Whence

\[ \frac{4 \beta - \sqrt{4 \beta^2 - 4k^2}}{2}, \quad \frac{4 \beta + \sqrt{4 \beta^2 - 4k^2}}{2}. \]

The roots of \( \mu^2 - 6\mu + 1 = 0 \) are \( 17158 \ldots \) and \( 582842 \ldots \)

As we suppose \( \mu \) less than 1, it follows that in no case must \( \mu \) exceed the lower root, \( 17158 \ldots \)

From (7) it follows that \( \beta > k \).

From (6) it follows that \( 1 - \mu > \mu'k' \), or \( = \mu'k'B \).

\[ \mu'k'B \text{ must not exceed } \frac{5}{4}. \]

Also \( \beta \) must not exceed \( \frac{3}{4} \).

To take Dr. Wallace's example, put \( \mu = 1 \).

We find, then, that \( kk' \) must not exceed \( 225 \). If we put \( kk' = 225 \), and solve for \( k \), we find that

\[ k = 342 \ldots \]

But \( kk' \) must not exceed \( 225 \); hence \( k \) must not lie between 342 and 658.

Take, for example, \( k = 2 \).

Then by (7), \( \beta = 225 \)

By (10) we must reject the second value of \( \beta \).

Adopting the first value, we find from equations (2) and (1) the following two solutions,

\[ x = 798 \ldots, \quad y = 104 \ldots, \quad z = 098 \ldots \]

\[ x = 308 \ldots, \quad y = 040 \ldots, \quad z = 652 \ldots \]

Here the effective fertility of the hybrids, inter se and with the parent varieties, must exceed 34 per cent. of that of the parent varieties; and in no case must it exceed 50 per cent. of the latter. Also, in no case must the parent species supply more than, or even as much as, 18 per cent. of its total progeny to the variant species.

C. If no hybrid unions occur, and the two varieties supply insignificant units to each other in such a way that, taking the progeny of the generation \( (x, y) \), a fraction \( \lambda \) of the \( x \) progeny belongs to the \( y \) variety, while a fraction \( \mu \) of the \( y \) progeny belongs to the \( x \) variety (where \( \lambda \) and \( \mu \) are proper fractions), it is easy to prove that in the ultimate, established, state of the total species

\[ x = 1 \lambda x + \lambda y + \mu y = \mu x + x + y = 1, \]

whence

\[ \lambda x = \mu y; \]

i.e.

\[ x : y = \mu : \lambda. \]

This may also be proved by direct calculation.

Woodroffe, Bournemouth.

J. W. Sharpe.