Empirical Test of Affine Stochastic Discount Factor Models of Currency Pricing

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In this paper I examine the affine model of currency pricing proposed by Backus, Foresi and Telmer (2001). Although they showed that affine models can reproduce the forward premium anomaly, I present evidence that these types of models deliver the anomaly at the cost of making assumptions inconsistent with the data. The model used by Backus, Foresi and Telmer assumes that conditional second moments are linear functions of the state variables. I find that this type of conditional heteroskedasticity is not supported by the data. When the heteroskedasticity assumption is relaxed, the model fails to reproduce the forward premium anomaly.

KEYWORDS: affine; SDF; exchange rates; currency pricing; forward premium anomaly

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1 Introduction

A number of papers in the international finance literature attempt to explain the negative correlation between the forward premium and exchange rate depreciation—a phenomenon known as “forward premium anomaly.” Fama (1984) in his well-known paper suggested that the anomaly occurs because the risk premium (the difference between forward rate and expected spot rate) varies over time and is negatively correlated with expected exchange rate depreciation. Backus, Foresi and Telmer (2001) (BFT) used Fama’s explanation to formulate an asset pricing model that accounts for the existence of the anomaly. They described a general framework for using stochastic discount factor (SDF) models to generate time-varying risk premium and to explain the forward premium anomaly. BFT also showed how a limited class of SDF models—the affine models—can generate the forward premium anomaly.

In this paper, I examine the affine model proposed by BFT. Although they showed that affine models can reproduce the forward premium anomaly, I present evidence that these types of models deliver the anomaly at the cost of making assumptions inconsistent with the data. Affine models assume that conditional second moments are linear functions of the state variables. The BFT model relies on this assumption to reproduce the forward premium anomaly. I find that this type of conditional heteroskedasticity is not supported by the data. I also find that, when this assumption is relaxed, the model fails to reproduce the forward premium anomaly. Therefore, I conclude that, to reproduce the anomaly, the BFT model relies on assumptions inconsistent with the data. I also present evidence that the data reject the assumption that the pricing kernels and

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1 A survey of the literature by Engel (1996) presents the nature of the anomaly from several angles and compares a large number of empirical tests that appeared in the literature.

2 For a reference on the stochastic discount factor models and the affine models, see Cochrane (2001) and Campbell, Lo and MacKinlay (1997)
state variables are conditionally log-normal – an assumption crucial for affine stochastic discount factor models.

These conclusions are based on the estimates of two versions of the BFT model for pound/dollar, mark/dollar and yen/dollar exchange rates. The first version is a single-state BFT model with a few minor modifications. I estimate this model and check whether it correctly characterizes conditional second moments by testing whether the difference between implied conditional second moments and their realizations is correlated with the state variable. If the second moments are modeled correctly, this difference should be i.i.d. I find that the null hypothesis of no correlation is rejected - the difference between the implied second moments and their realizations is not i.i.d. At the same time, I find that parameter estimates reproduce the forward premium anomaly.

In the second model, I introduce more substantial modification to the BFT model. The second model is a two-state affine model with one heteroskedastic and one homoskedastic state variable. The main point of this modification is to allow variables (forward premium and exchange rate depreciation) to change without affecting conditional second moments. This version of the model also can deliver the forward premium anomaly. I conduct the same tests as in the single-state version of the model and find that, for all three exchange rates, their conditional moments are characterized correctly – i cannot reject the hypothesis that the difference between the implied second moments and their realizations is i.i.d. However, the estimates imply that the anomaly does not exist. Hence, I conclude that the model can either correctly characterize conditional second moments or deliver the forward premium anomaly but it cannot do both at the same time.

With this paper I intend to contribute to a recent area of research on the forward premium anomaly that stems from asset pricing literature. Because financial markets appear to be inte-
grated across developed countries (thus, agents should have equal access to domestic and foreign assets), one can apply the existing asset pricing literature with few adjustments for currency pricing. The BFT model is an application of the stochastic discount factor model introduced by Cox, Ingersoll and Ross (1985).

The remainder of the paper is organized in the following way. In the next section, I describe the nature of the anomaly and how stochastic discount factor models can be applied to study this issue. In the third section, I present the BFT affine model in the general form and discuss some of its implications. The fourth section describes the single-state model and presents its estimation results. In the fifth section, I extend the model to include two state variables. Then, I summarize and discuss the results presented in this paper in the conclusion.

2 Affine Models of Currency Pricing

2.1 Forward Premium Anomaly

The centerpiece of the literature on the forward premium anomaly likely is the following regression:

$$\Delta s_{t+1} = \alpha_1 + \alpha_2 (f_t - s_t) + \varepsilon_t,$$

where $s_t$ and $f_t$ are logs of the spot and the forward exchange rates defined in dollars per unit of foreign currency. Coefficient $\alpha_2$ sometimes is referred to as the Fama coefficient, named for Fama (1984). The overwhelming majority of studies find $\alpha_2$ negative and significantly different from zero. Most studies find at least some evidence of the anomaly by showing that $\alpha_2$ is more than two standard errors away from 1, which is interpreted as a failure of the hypothesis that forward rate is an unbiased predictor of the spot rate.
The sample used in this paper covers the period from January 1975 through December 2001. Least squares regression estimates of equation 1 based on this sample also show that $\alpha_2$ is negative and significantly different from 0 for yen/dollar exchange rate and significantly different from 1 for the pound/dollar and mark/dollar exchange rates (shown in table 1.)

To explain why the anomaly occurs, Fama (1984) decomposes the forward premium into the risk premium $p_t$ and the expected rate of depreciation $q_t$:

$$f_t - s_t = f_t - E_t s_{t+1} + (E_t s_{t+1} - s_t) \equiv p_t + q_t.$$ 

This decomposition allows to write the OLS estimator of $\alpha_2$ as

$$\alpha_2 = \frac{Var(p+q)}{Var(p+q)} = \frac{Cov(q,p) + Var(q)}{Var(p+q)}.$$

This expression shows that, for a model to be able to reproduce the forward premium anomaly, two conditions must hold: a) Covariance of $p_t$ and $q_t$ must be negative and b) The absolute value of the covariance of $p_t$ and $q_t$ should exceed the variance of $q_t$. Thus, time-variant risk premium is crucial to model’s ability to reproduce the anomaly.

### 2.2 Applying Stochastic Discount Factor Models to Currency Pricing

The intuition for using stochastic discount factor models for currency pricing presented here closely follows BFT. First, assume there exist two economies with integrated financial and currency markets – a domestic economy, whose currency is denominated in dollars, and a foreign economy with its currency denominated in pounds. Then, allowing no pure arbitrage opportunities between these economies, the pricing relations for each of these economies can be derived.
as

$$v_t = E_t [d_{t+1} M_{t+1}] ,$$

(2)

where \(v_t\) is the dollar value of the state contingent claim \(d_{t+1}\), and \(M_{t+1}\) is the stochastic discount factor or pricing kernel.

For the assets denominated in pounds, one can write a similar expression:

$$v_t^* = E_t [d_t^* M_t^*] ,$$

where superscript asterisk denotes variables pertinent to the foreign economy. If the financial markets are fully integrated, pound denominated assets can be priced using the dollar pricing kernel. Using this notation, the pricing relation for foreign assets using the domestic pricing kernel can be expressed in the following way:

$$v_t^* = E_t \left[ d_t^* \left( \frac{S_{t+1}}{S_t} \right) M_t^* \right] ,$$

where \(S_t\) represents the spot exchange rate at time \(t\).

If there are no arbitrage opportunities, it must also be true that

$$E_t [d_{t+1} M_{t+1}^*] = E_t \left[ d_{t+1}^* \left( \frac{S_{t+1}}{S_t} \right) M_{t+1} \right]$$

(3)

because the price of the asset should be the same regardless of which kernel is used.

As BFT, this relation is satisfied when \(M_{t+1} \left( \frac{S_{t+1}}{S_t} \right) = M_t^*\), and this solution is unique if asset and currency markets are complete. When markets are incomplete, the choices of \(M\) and \(M^*\) satisfying equation 2 are not unique. However, as BFT noted, one still could choose them to satisfy equation 3. This equation ties two markets together by describing three stochastic processes and makes one of them redundant. Specifying stochastic behavior of \(M\) and \(M^*\) also determines how \(\frac{S_{t+1}}{S_t}\) should behave.
Therefore, this result can be used to write the log of depreciation of the exchange rate in the following form:

\[ m_{t+1}^* - m_{t+1} = s_{t+1} - s_t, \]  

(4)

where \( m \) is the log of the pricing kernel.

Using this result, the expression for the forward premium can be derived from kernel equations. Consider a forward contract at period \( t \), to exchange one pound into \( F_t \) dollars at time \( t + 1 \), where \( F_t \) is the forward rate. This contract requires no payment at time \( t \), so \( v_t = 0 \). Therefore, the net cash flow in dollars at time \( t + 1 \) is just \( F_t - S_{t+1} = d_{t+1} \). Thus, equation 2 can be written in the following way:

\[ 0 = E_t [(F_t - S_{t+1}) M_{t+1}] \]

Dividing by \( S_t \) and using equation 3, obtain

\[ \left( \frac{F_t}{S_t} \right) E_t (M_{t+1}) = E_t \left( M_{t+1} \frac{S_{t+1}}{S_t} \right) = E_t (M_{t+1}^*). \]

Taking the log of the last equation, gives the expression for the forward premium in terms of pricing kernels:

\[ f_t - s_t = \log E_t (M_{t+1}^*) - \log E_t (M_{t+1}) \]  

(5)

Until this point, no assumptions have been made about stochastic behavior of the pricing kernels. So, to close the model, the next step is to use affine models to parametrize the stochastic process for the pricing kernel. In the following sections I present the general form of the BFT model with an arbitrary number of state variables. Then, I describe and show the estimates for the single-factor and the two-factor versions of the model.
3 BFT Model

3.1 Setup

The model has two key components: state equations that describe the stochastic process of the state variables and kernel equations that describe how the pricing kernel depends on the state variable. The BFT model specifies state equations in the following way:

\[ z_{t+1} = (I - \Phi) \theta + \Phi z_t + V(z_t)^\frac{1}{2} \varepsilon_{t+1}, \]  

(6)

where \( z_t, \varepsilon_{t+1} \) are vectors of state variables and innovations respectively, and the innovation \( \varepsilon_{i,t+1} \) is a standard normal random variable. \( V(z_t) \) is diagonal with \( v_{ii} = \alpha_i + \beta_i z_{i,t} \). Matrix \( \Phi \) is stable with positive roots. Vector \( \theta \) contains only positive elements. This setup does not rule out that \( z \) can become negative but the probability of that becomes smaller with shorter time intervals. In continuous time, when Feller condition\(^4\) is satisfied, this setup guarantees that \( z \) remains positive.

For convenience, the kernel equations are written in terms of the negative of the log of the pricing kernel:

\[ -m_{t+1} = \delta + \gamma' z_t + \lambda' V(z_t)^\frac{1}{2} \varepsilon_{t+1} + \eta_{t+1}, \]  

\[ -m_{t+1}^* = \delta^* + \gamma^* z_t + \lambda^* V(z_t)^\frac{1}{2} \varepsilon_{t+1} + \eta^*_{t+1}, \]  

(7)

where \( \eta \) is also a normal random variable \( \eta \sim N(0,\sigma^2) \).\(^5\)

\(^4\)See BFT for a detailed discussion of this issue.

\(^5\)The structure of the kernel equations shown here is slightly different from the set up by BFT. In the BFT model, innovations to the pricing kernel depend only on the innovations to the state variable times a scalar – there is no i.i.d. innovation \( \eta \). This random disturbance term allows for an additional source of innovations to the pricing kernel (and exchange rates) that does not have any impact on contemporaneous or future short rates. Campbell et. al (1997, Ch. 11) discuss a similar setup of the affine model as we present here. I include
Next, derive the solution for short rates (i.e., the yield on one-period bond) using state and kernel equations. From equation 2 one can derive the price of the risk-free one period discount bond \( v_t \) that pays one dollar \((d_t = 1)\) at date \( t + 1 \). Expressing in logs, one can write the short rate \( r_t \) as logarithm of that bond’s price:

\[
 r_t = -\log v_t = -\log E_t [M_{t+1}]. \tag{8}
\]

Affine models describe the solution for the short rates as a linear function of the state variables:

\[
 r_t = -\log E_t [M_{t+1}] = -(E_t [m_{t+1}] + \frac{1}{2} Var_t [m_{t+1}]) = A + Bz_t,
\]

where \( A \) and \( B \) are functions of parameters of the state and kernel equations (derived in the appendix). Similarly one can find the short rate for the foreign economy as \( r^*_t = A^* + B^*z_t \).

Recall from equation 5 that the forward premium can be expressed in terms of the log of expectation of the pricing kernel. Therefore, the forward premium also can be expressed in terms of the state variables:

\[
 f_t - s_t = \log E_t M^*_{t+1} - \log E_t M_{t+1} = r_t - r^*_t = A - A^* + (B - B^*)z_t. \tag{9}
\]

this innovation term because, without it, in the single-state model the exchange rate depreciation must exhibit as much persistence as the forward premium. Table 1 shows that this is not true—autocorrelation of the exchange rate depreciation is close to zero, and the forward premium is very persistent.
From this equation it can be seen that the model implies that the forward premium is a linear function of the state variables.

Depreciation of the spot exchange rate also can be expressed in terms of the state variables. From equations 4 and 7 exchange rate depreciation is

$$\Delta s_{t+1} = (\delta - \delta^*) - (\gamma - \gamma^*)z_t + (\lambda - \lambda^*)V(z_t)^{1/2} \varepsilon_{t+1} + \eta_{t+1} - \eta^*_{t+1}.$$  (10)

### 3.2 BFT Model and the Forward Premium Anomaly

To reproduce the forward premium anomaly, the model must be able to generate negative covariance between exchange rate depreciation and the forward premium. In the BFT model, this covariance is

$$Cov(\Delta s_{t+1}, f_t - s_t) = [(\gamma - \gamma^*) - (\tau - \tau^*)]V(z_t)(\gamma - \gamma^*).$$  (11)

where $\tau_i = \lambda_i^2 \beta_i / 2$ and $\tau^*_i = \lambda_i^2 \beta^*_i / 2$. The last equation conveys the intuition behind the BFT model. To reproduce the anomaly, state variables must affect pricing kernels asymmetrically: There has to be at least one state variable that has a greater effect on the level of one kernel (e.g. $\gamma > \gamma^*$) and the variance of another ($\tau < \tau^*$). As BFT noted, “if $V(z_t)$ is diagonal, elements of $[(\gamma - \gamma^*) - (\tau - \tau^*)]$ and $(\gamma - \gamma^*)$ must have different signs for at least one $i$.”

### 3.3 Implication of the BFT Model

When the BFT model reproduces the forward premium anomaly, it carries several implications for stochastic behavior of the forward premium and the exchange rate depreciation.

1. **Conditional variance of the exchange rate depreciation** $Var_t(\Delta s_{t+1})$ must be correlated with the forward premium $(f_t - s_t)$. In terms of the model’s parameters, conditional variance of
exchange rate depreciation can be expressed as a linear function of the state variables as

\[ Var_t (\Delta s_{t+1}) = (\lambda - \lambda^*)' V (z_t) (\lambda - \lambda^*) + Var (\eta - \eta^*). \] (12)

The anomaly requires that \((\tau - \tau^*) \neq 0\), which implies that \((\lambda - \lambda^*) \neq 0\). Therefore, conditional variance of \(\Delta s_{t+1}\) must be correlated with the forward premium. In the single-state model, conditional variance of \(\Delta s_{t+1}\) and the forward premium are perfectly correlated.

2. **Conditional variance of the forward premium** \(Var_t (f_{t+1} - s_{t+1})\) **is correlated with the forward premium** \((f_t - s_t)\). Equation 9 shows that the forward premium is a linear function of the state variables. If all state variables are conditionally heteroskedastic, so must be the forward premium. In a model with a single state variable, conditional variance \(Var_t (f_{t+1} - s_{t+1})\) is perfectly correlated with forward premium.

3. When the model reproduces the forward premium anomaly, **innovations to the exchange rate depreciation and forward premium must be correlated. Their covariance is correlated with the forward premium.** Innovation to the exchange rate depreciation is

\[ \psi_{t+1} \equiv (\Delta s_{t+1}) - E_t (\Delta s_{t+1}) = (\lambda - \lambda^*) V (z_t) \frac{1}{2} \epsilon_{t+1} + \eta_{t+1} - \eta^*_{t+1}. \]

Innovation to the forward premium is

\[ \xi_{t+1} \equiv (f_{t+1} - s_{t+1}) - E_t (f_{t+1} - s_{t+1}) = (\gamma - \gamma^*) V (z_t) \frac{1}{2} \epsilon_{t+1}. \]

Thus, covariance between these two innovations is

\[ Cov_t (\psi_{t+1}, \xi_{t+1}) = (\lambda - \lambda^*)' V (z_t) (\gamma - \gamma^*). \]

The anomaly implies that at least one pair of elements in \((\lambda - \lambda^*)\) and \((\gamma - \gamma^*)\) must be nonzero. Therefore, conditional covariance is also correlated with the forward premium.
In this paper, I conduct statistical tests of these implications. I also examine whether the BFT model places restrictions on other moments such as autocorrelation of the forward premium, autocorrelation of exchange rate depreciation and the mean of exchange rate depreciation. I discuss those moments only in the context of single-state and two-state models.

4 Single-State Model

4.1 Setup of the model.

In the previous section, I presented the general setup of the BFT model. Now I show how I parametrize the single-state model. The setup of the single state model is similar to the single-state example shown in BFT. The state and kernel equations are

\[ z_{t+1} = (1 - \phi) \theta + \phi z_t + \sigma z_t^2 \varepsilon_{t+1}, \]  

\[ m_{t+1} = \delta + (\gamma + \frac{\lambda^2}{2}) z_t + \lambda z_t^2 \varepsilon_{t+1} + \eta_{t+1}, \]  

\[ m^*_{t+1} = \delta^* + (\gamma^* + \frac{\lambda^2}{2}) z_t + \lambda^* z_t^2 \varepsilon_{t+1} + \eta^*_{t+1}. \]

The forward premium and the exchange rate depreciation equations now look as follows:

\[ f_t - s_t = (\delta - \delta^*) - \frac{\sigma^2 - \sigma^2^*}{2} - (\gamma - \gamma^*) z_t, \]  

\[ \Delta s_{t+1} = (\delta - \delta^*) - \left( \gamma - \gamma^* + \frac{\lambda^2 - \lambda^2}{2} \right) z_t + (\lambda - \lambda^*) z_t^2 \varepsilon_{t+1} + \eta_{t+1} - \eta^*_{t+1}. \]

\[ \text{The single-state version of the model has an apparent flaw because it implies that the short rates in both economies, one-month forward premium and longer term forward premium are all perfectly correlated. This is not true. For example, one-month and three-month forward premia are highly but not perfectly correlated. If this were the only flaw of the model one could conclude that the model performs well overall. However, I use this single-state model to show that the model fails in other aspects as well.} \]
The Fama coefficient is now $\alpha_2 = 1 + \frac{\lambda^2 - \lambda^*}{2(\gamma - \gamma^*)}$, and the conditional variance of exchange rate depreciation is

$$\text{Var}_t (\Delta s_{t+1}) = (\lambda - \lambda^*)^2 z_t + \text{Var} (\eta - \eta^*) .$$

(17)

It’s easy to derive the stochastic process for the forward premium because it is just a linear function (equation 15) of the state variable:

$$f_{t+1} - s_{t+1} = \beta + \phi (f_t - s_t) + (a + b (f_t - s_t))^2 \varepsilon_{t+1} .$$

(18)

where $\beta, a$ and $b$ are functions of model parameters. Thus, conditional variance of the forward premium is perfectly correlated with the lagged forward premium.

When I estimate the model, I also check if the model reproduces other moments of the data. The moments I examine are autocorrelation of the forward premium (which is the autoregressive parameter $\phi$ from the state equation); autocorrelation of the exchange rate depreciation; and the mean of exchange rate depreciation (shown in the appendix). Sample estimates of these three moments presented in table 1 show high autocorrelation in the forward premium and low autocorrelation in exchange rate depreciation. I compute implied moments from parameter estimates and compare them with the sample estimates.

Because the existence of the forward premium anomaly can be attributed to the time-varying risk premium, it is useful to derive the expression for the risk premium $p_t$ implied by the model:

$$p_t = f_t + s_t - (E_t \Delta s_{t+1})$$

$$= -\frac{\lambda^2 - \lambda^*}{2} z_t + \left( \frac{\sigma^2_\eta - \sigma^2_{\eta^*}}{2} \right) .$$

The risk premium has two components. The first component is a time varying risk premium that depends on the relative difference of the $\lambda$'s. These $\lambda$'s govern the slopes of the yield curves.
in each economy. Hence, the first component of the risk premium captures the intuition that the forward risk premium is determined by the relative differences in pricing of interest rate risks across countries. The second component of the risk premium does not affect the slope of the yield curve and, hence, represents risks other than interest rate risks. In other words, changes in conditional moments of the forward risk premium are due entirely to interest rate risks, but the overall level of risk premium also depends on risks other than interest rate risks.

4.2 Estimation Results

The estimation results for this section are presented in table 2 in the appendix. Not all of the parameters in the model can be identified. The issue of identification and the estimation procedure is detailed in the appendix. The lower part of table 2 shows that, in all cases, the model successfully reproduces the forward premium anomaly. In all cases, estimates of the implied Fama coefficient are very close to the OLS estimates of equation 1. The implied Fama coefficient is significantly negative in all three cases.

Parameter \((\gamma - \gamma^*)\) measures the difference of the influence of the state variable on the short rates in the foreign and domestic economy. The estimates show that the difference is positive and significant in all cases. Reduced form parameter \(\frac{\lambda^2 - \lambda^{*2}}{2}\) represents the difference in the effect of the state variable on the conditional variances of the pricing kernels. Its estimates are negative and significant in all cases. These results support the idea behind the BFT model that the anomaly is due to the asymmetric effects of the state variable on the level and the conditional variance of the pricing kernels.

I now turn to testing other implications of the model. For that I construct innovations \(\psi_{t+1}\) and \(\xi_{t+1}\) defined earlier. Using these innovations I construct realizations of the conditional second
moments. The square these innovations is the realization of the variances and the product of $\psi_{t+1}$ and $\xi_{t+1}$ is the covariance. Then, I regress the difference between the realizations of the second moments and their conditional expectations on the forward premium. For example, for the exchange rate depreciation, I estimate the following regression:

$$\psi_{t+1}^2 - \text{Var}_t(\Delta s_{t+1}) = \alpha_0 + a_1 (f_{1,t} + s_t) + \nu_t,$$

where $\text{Var}_t(\Delta s_{t+1})$ is the expected variance implied by the parameters estimates. If the model correctly characterizes conditional second moments then regression intercept and slope coefficients must not be significantly different from zero. Estimates of those coefficients are shown in table 3. These results show that, for each exchange rate, at least one of the these conditional moments is not characterized correctly. In the case of the pound/dollar exchange rate, the estimates reject characterization of the conditional covariance and conditional variance of the forward premium.

For the mark/dollar rate, conditional variance of forward premium is modeled incorrectly. For the yen/dollar rate conditional variance of both the exchange rate depreciation and the forward premium are modeled incorrectly.

I also examine whether the model reproduces other moments of the data. I check the autocorrelation of the forward premium and exchange rate depreciation and the mean of the forward premium. The lower panel of table 2 contains the estimates implied by the parameters of the model and the sample estimates. These results show that the moments implied by parameter estimates are very close to the sample moments. The parameters of the model imply in all cases that the null hypothesis that $E[\Delta s_{t+1}] = 0$ cannot be rejected. Parameter estimates also imply that the autocorrelation of $\Delta s_{t+1}$ is smaller then the simple estimate from the sample. Autoregressive parameters $\phi$ (which are also autocorrelation coefficients for the forward premia) are between 0.70 and 0.89 and statistically significant, indicating high persistence in the forward
premium. That is consistent with the sample estimates.

Affine models rely on assumptions of lognormality of variables to derive many of their results. Table 4 shows the sample moments of standardized $\varepsilon_t$ and $\eta_t - \eta^*_t$. High fourth moments imply that there are many observations far in the tails of the distribution.

Thus, results presented in this section show that this simple single-state model performs quite well when performance criteria include only unconditional moments such as Fama regression coefficient, mean of exchange rate depreciation and autocorrelations of forward premia and exchange rate depreciation. However, when I test conditional second moments I find that they are not modelled correctly. Notably, the way these conditional moments are specified is crucial for the model’s ability to reproduce the forward premium anomaly.

5 Two-Factor Model

5.1 Setup of the Model

In this section, I modify the single-state model so it still can reproduce the forward premium anomaly without necessarily implying the restrictions rejected in the previous section. I add another state variable to the model, and this state variable is homoskedastic. What I hope to accomplish by this is to decompose the forward premium into two components: one of which has an effect on the conditional variances and one of which doesn’t. The setup of the model is as follows:

$$-m_{t+1} = \delta + \left( \gamma + \frac{\lambda^2}{2} \right) z_t + \mu x_t + \lambda z_t^2 \varepsilon_{t+1} + \eta_{t+1}, \quad (19)$$

$$-m^*_{t+1} = \delta^* + \left( \gamma^* + \frac{\lambda^*_2}{2} \right) z_t + \lambda^* z_t^2 \varepsilon_{t+1} + \eta^*_t, \quad (20)$$

$$z_{t+1} = \theta z (1 - \phi z) + \phi z z_t + \sigma z z_t^2 \varepsilon_{t+1}, \quad (21)$$
\[ x_{t+1} = \theta_x (1 - \phi_x) + \phi_x x_t + \sigma_x \zeta_{t+1}. \] (22)

The two state variables are now \( x_t \) and \( z_t \). Subscripts that now appear on the parameters of the state equations ascribe the parameters to the relevant equations. As before, innovations are assumed to be uncorrelated with each other and normally distributed. The state variable \( x_t \) appears only in one of the kernel equations. I arbitrarily choose the domestic kernel. The choice is innocuous because I are working with the forward premium, not the interest rates – regardless of which kernel equation I modify (or if I include the variable in both equations), the forward premium and the exchange rate depreciation have the same reduced form. The general representation of the BFT model (equations 6 and 7) nests this two-factor model. However, I am considering a stochastic process for the state variable that can no longer guarantee implied bond yields to be always positive.

The expressions for the exchange rate depreciation and the forward premia are

\[ \Delta s_{t+1} = (\delta - \delta^*) - \left( \gamma - \gamma^* + \frac{\lambda^2 - \lambda^{*^2}}{2} \right) z_t + \mu x_t + (\lambda - \lambda^*) z_{t+1}^\frac{1}{2} \tilde{\varepsilon}_{t+1} + \eta_{t+1} - \eta^*_{t+1}, \] (23)

\[ f_{n,t} - s_t = (A_n - A_n^*) - (B_n - B_n^*) z_t + C_n x_t, \] (24)

where capital letters represent the affine coefficient derived in appendix. Notice a new subscript \( n \) in the notation – it denotes the number of periods to maturity. I add this to the notation because I use one-month and three-month forward premia to estimate the model and I need to be able to distinguish between these two variables.

Adding a second state variable changes the way the model characterizes second conditional moments:

1. The forward premium is a function of two state variables and only one of them is heteroskedastic. Thus, the model no longer implies that conditional variance of the forward
premium must change with its level.

2. Conditional variance of the exchange rate depreciation now depends only in part on the forward premium – it is no longer perfectly correlated with the forward premium. The forward premium can change due to change in the state variable $x$, and it would not affect $Var_t (\Delta s_{t+1})$.

3. Similarly, innovations $\xi_{t+1}$ and $\psi_{t+1}$ are still correlated, but their covariance is no longer perfectly correlated with the forward premium.

The Fama coefficient is now

$$Fama = 1 + \frac{(B_1 - B_1^*) \left( \frac{\lambda^2 - \lambda^{*2}}{2} \right) Var(z_t)}{(B_1 - B_1^*)^2 Var(z_t) + C_1^2 Var(x_t)}.$$ 

Thus, the model still is capable of reproducing the forward premium anomaly, if the numerator is negative and large enough.

The expression for the risk premium looks the same way as in the single-factor case:

$$p_t = f_t - s_t - (E_t \Delta s_{t+1})$$

$$= -\frac{\lambda^2 - \lambda^{*2}}{2} z_t + \left( \frac{\sigma_0^2 - \sigma^{*2}}{2} \right).$$

As before, the risk premium has two components. The first component is a time varying risk premium that depends on the relative difference of the $\lambda$'s. However, notice that in the single-factor setup, the forward premium and the risk premium were perfectly correlated because they both were linear functions of the same state variable. Here they are not perfectly correlated because now the forward premium is a function of two state variables. Thus, by adding a homoskedastic state variable we can decompose the forward premium into two factors – one that
affects risk premium and one that doesn’t.

5.2 Estimation Results

Estimation results are presented in table 5. The top panel of the table shows parameter estimates, and the bottom part shows moments implied by those parameters. I find that, in all cases, the model does not reproduce the forward premium anomaly. Implied Fama coefficients are between 0.77 and 1.23. The standard errors are small enough to reject the null hypothesis that the implied coefficients are negative.

Table 6 presents the result of the specification tests of the model. As in the single-state version, I compute the innovations and test whether the difference between realizations and the expectations of conditional second moments is i.i.d. The results show one cannot reject the hypothesis that the model correctly characterizes conditional moments for all three exchange rates.

The estimates of the autoregressive parameters $\phi_z$ and $\phi_x$ are another interesting result for this model. These estimates show that the homoskedastic state $x_t$ appears very persistent ($\phi_x$ is between 0.93 and 0.97), and the heteroskedastic state shows little autocorrelation ($\phi_z$ is between 0 and 0.17.) To see which state variable explains more variation in the forward premia, I regressed forward premia on the second state variable. The $R^2$ of these regressions are 0.87, 0.88 and 0.74 for pound, mark and yen exchange rates respectively. The homoskedastic state variable explains most of the variation in the forward premia (sample correlations between the state variables are 0.03 or less. Therefore, regression on the heteroskedastic variable would be a mirror image of this regression, because forward premia are exact functions of the two state variables). This might explain why the forward premium anomaly is not reproduced by this model. Heteroskedasticity is
essential for the model’s ability to reproduce the anomaly, and stochastic behavior of the forward premium for these three exchange rates is best explained by a homoskedastic state variable.

The two-state version of the model does a good job reproducing other sample moments. Autocorrelation of the forward premium and the exchange rate depreciation implied by the parameter estimates (presented in the lower panel of table 5) are close to the sample estimates.

Table 7 shows the sample moments of standardized $\varepsilon_t$, $\zeta_t$, and $\eta_t - \eta^*_t$. The table shows that fourth moments in the sample are still very high – the innovations are not normally distributed.

Thus, the estimates of the two-factor version of the model show that it fails to reproduce the Fama coefficient for all three countries. Including a homoskedastic state variable into the model relaxes several implications of the model that have been rejected in the single-state version. The tests show that with two factors one cannot reject the hypothesis that the conditional second moments are modeled correctly. Therefore, it appears that the model is not capable of delivering the forward premium anomaly and correctly characterizing conditional second moments.

6 Conclusion

In this paper I explore affine models of currency pricing similar to BFT. I find that the single-state version of the BFT model successfully reproduces many features of the data, including the forward premium anomaly. When I estimate the model using the data on three exchange rates, I find that the model implies Fama coefficients that are very close to their OLS estimates. Implied autocorrelations of forward premium and exchange rate depreciation, as well as the mean of exchange rate depreciation, are close to their sample counterparts.

However, when I test the implications of the model for conditional second moments, I find that they are rejected. I test whether the model correctly characterizes conditional variance of
the forward premium and exchange rate depreciation and conditional covariance of these two variables. The results show that for each exchange rate at least on of these moments is not modeled correctly.

When I modify the model by including a homoskedastic state variable, I relax the implications for the conditional moments. With the two-state model, variation in the forward premium can be decomposed into two factors – the heteroskedastic factor that affects conditional moments and the risk premium and the homoskedastic factor that affects only levels of the forward premium and the exchange rate depreciation. The two-state model still is capable of reproducing the anomaly as well as other moments in the data. However, when I estimate this version of the model, I find it does not reproduce the anomaly. On the other hand, I can no longer reject the hypothesis that the conditional second moments are not modeled correctly. I also find that the homoskedastic state explains most variation in the forward premia. Therefore, the evidence suggests that the model is unable to reproduce the anomaly and correctly specify conditional moments simultaneously.

The object of the empirical investigations I perform in this paper is to derive the most parsimonious representations of SDF model that reflects most features of the data. The results presented in this paper suggest that affine models with few state variables might not be able to reflect some of the features of the data. This could be solved potentially with a model that has more state variables, but those are difficult to estimate. Thus, a possible solution to the problem could lie outside of the class of affine models. Bansal (1997) developed a general framework of stochastic discount factor models that nests affine models. He showed that it is possible to reproduce the forward premium anomaly with a symmetric (state variables affect countries in the same way) model that does not imply a linear relationship between the conditional variance
of the forward exchange rates and forward premium.

This paper can be extended in two possible directions. It appears that one needs a stochastic discount factor model that does not rely on a linear relationship between the state variables and the conditional moments to reproduce the forward premium anomaly and characterize conditional second moments. One way to introduce nonlinearity into the model is to change the way the state variable affects the conditional variance of the innovations. A potential candidate is a quadratic term structure model such as described in Constantinides (1992) or Ahn et al. (2002). Another possible direction is to formulate a switching process, that would allow for time-varying risk premia. This switching model could be capable of reproducing the forward premium anomaly while fitting other features of the data without implying linear relationship between the state variables and conditional moments.

References


Fama E., 1984, Forward and Spot Exchange rates, Journal of Monetary Economics, 14, 319-338

Hodrick R., Vassalou M., 2000, Do We Need Multi-country Models to Explain Exchange Rate and Interest Rate Dynamics? Columbia University working paper.

A Appendix

A.1 Deriving The Parameters of the Affine Model

Here I show how the parameters $A_1, A_3, B_1, B_3, C_1$ and $C_3$ are derived. I start by rewriting the kernel and the state equations:

$$-m_{t+1} = \delta + \left( \gamma + \frac{\lambda^2}{2} \right) z_t + \mu x_t + \lambda z_t^{\frac{1}{2}} \varepsilon_{t+1} + \eta_{t+1},$$

$$-m_{t+1}^* = \delta^* + \left( \gamma^* + \frac{\lambda^2}{2} \right) z_t + \lambda^* z_{t+1}^{\frac{1}{2}} \varepsilon_{t+1} + \eta_{t+1}^*,$$

$$z_{t+1} = \theta_z (1 - \phi_z) + \phi_z z_t + \sigma_z z_t^{\frac{1}{2}} \varepsilon_t,$$

$$x_{t+1} = \theta_x (1 - \phi_x) + \phi_x x_t + \sigma_x \xi_t.$$

From these equations one can find the domestic short rate:

$$r_{t+1} = -\log E_t M_{t+1} = -E_t m_{t+1} - \frac{1}{2} Var(m_t)$$

$$= \delta + \gamma z_t + \mu x_t - \frac{\sigma_x^2}{2}.$$
And, similarly, the foreign short rate:

\[ r^{*}_{1,t} = \delta^* + \gamma^* z_{1,t} - \frac{\sigma^2_\mu^2}{2}. \]

To get the solution for the three-period interest rate, guess \( r_{n,t} = A_n + Z'_t B_n \).

It has to be true that

\[ p_{n,t} = E_t [\log m_{t+1} + p_{n-1,t+1}] + (1/2) \text{Var}_t [\log m_{t+1} + p_{n-1,t+1}], \]

where \( p_{n,t} \) is the price at time \( t \) of the bond maturing at \( t + n \). Consider the components of this expression separately:

\[
E_t [\log m_{t+1} + p_{n-1,t+1}] = -\delta - \left( \gamma + \frac{\lambda^2}{2} \right) z_t - \mu x_t - A_{n-1} - B_{n-1} (1 - \phi_z) \theta_z - B_{n-1} \phi_z z_t - C_{n-1} (1 - \phi_x) \theta_x - C_{n-1} \phi_x x_t,
\]

\[
\text{Var}_t [\log m_{t+1} + p_{n-1,t+1}] = \lambda^2 z_t + B_{n-1}^2 \sigma_z^2 z_t + 2B_{n-1} \sigma_z \lambda z_t + C_{n-1}^2 \sigma_x^2 x_t + \sigma^2_\eta.
\]

Therefore,

\[
-p_{n,t} = \left( B_{n-1} \phi_x + \gamma - \frac{1}{2} B_{n-1} \sigma_x^2 - B_{n-1} \sigma_x \lambda \right) z_{1,t} + C_{n-1} \phi_x + \mu - \frac{1}{2} C_{n-1} \sigma_x^2 \right) x_t + \delta + A_{n-1} + B_{n-1} \theta_z (1 - \phi_z) + C_{n-1} \theta_x (1 - \phi_x) - \frac{1}{2} \sigma^2_\eta
\]

From the last expression one can get the solution for \( A_n \) and \( B_n \) and \( C_n \)

\[ A_n = \delta + A_{n-1} + B_{n-1} \theta_z (1 - \phi_z) + C_{n-1} \theta_x (1 - \phi_x) - \frac{1}{2} \sigma^2_\eta, \]
\[ B_n = B_{n-1}\phi + \gamma - \frac{1}{2}B_{n-1}^2\sigma_x^2 - B_{n-1}\sigma_x\lambda, \]
\[ C_n = C_{n-1}\phi + \mu - \frac{1}{2}C_{n-1}^2\sigma_x^2. \]

\( B_3, C_3 \) and \( A_3 \) can be found solving recursively and using the starting values \( B_1 = \gamma, C_1 = \mu. \)

### A.2 Expressions for Unconditional Moments in Terms of Parameters of the Model

In the single-state model autocorrelation can be found as
\[
\text{Corr}(\Delta s_{t+1}, \Delta s_t) = \frac{(\gamma - \gamma^* + \frac{\lambda^2 - \lambda^{*2}}{2})^2 \phi \theta \sigma^2}{\lambda - \lambda^*} + \left(\gamma - \gamma^* + \frac{\lambda^2 - \lambda^{*2}}{2}\right) \left(\lambda - \lambda^*\right) \theta.
\]

The mean of exchange rate depreciation is
\[
E(\Delta s_{t+1}) = (\delta - \delta^*) + \left[\gamma - \gamma^* + \frac{\lambda^2 - \lambda^{*2}}{2}\right] \theta.
\]

By construction, the latter expression should not be restricted to either negative or positive values, because that would imply ever-diverging exchange rates.

Next, I derive these moments for the two-state model.

- Autocorrelation of the forward premium

\[
\text{Corr}(f_{t+1} - s_{t+1}, f_t - s_t) = \frac{\text{Cov}(f_{t+1} - s_{t+1}, f_t - s_t)}{\text{Var}(f_t - s_t)}
\]
\[
= \frac{\mathcal{B}^2 \text{Cov}(z_{t+1}, z_t) + \mathcal{C}^2 \text{Cov}(x_{t+1}, x_t)}{\mathcal{B}^2 \text{Var}(z_t) + \mathcal{C}^2 \text{Var}(x_t)}
\]
\[
= \frac{\mathcal{B}^2 \phi z \theta \sigma^2}{\mathcal{B}^2 \theta \sigma^2 + \mathcal{C}^2 \phi z \sigma^2}
\]

where \( \mathcal{B}_n = (B_n - B^*_n) \). Depending on the relative sizes of the components the model should be able to accommodate any values \( 0 < \text{Corr}(f_t - s_t, f_{t-1} - s_{t-1}) < 1 \).
• Autocorrelation of $\Delta s_{t+1}$ is

$$
Corr(\Delta s_t, \Delta s_{t-1}) = \frac{Cov(\Delta s_t, \Delta s_{t-1})}{Var(\Delta s_t)}.
$$

Consider first the numerator of the above expression:

$$
Cov(\Delta s_{t+1}, \Delta s_t) = \left(B_1 + \frac{\lambda^2 - \lambda^*}{2}\right)^2 Cov(z_{t+1}, z_t) + C_1^2 Cov(x_{t+1}, x_t)
+ \left(B_1 + \frac{\lambda^2 - \lambda^*}{2}\right)(\lambda - \lambda^*) Cov\left(z_{t+1}, z_t \hat{\varepsilon}_{t+1}\right)
= \left(B_1 + \frac{\lambda^2 - \lambda^*}{2}\right)^2 \phi_z \frac{\theta_z \sigma_z^2}{1 - \phi_z^2} + C_1^2 \phi_x \frac{\sigma_x^2}{1 - \phi_x^2} + \left(B_1 + \frac{\lambda^2 - \lambda^*}{2}\right)(\lambda - \lambda^*) \theta_z \sigma_z.
$$

The variance of the exchange rate depreciation is

$$
Var(\Delta s_{t+1}) = \left(B_1 + \frac{\lambda^2 - \lambda^*}{2}\right)^2 Var(z_t) + C_1^2 Var(x_t) + (\lambda - \lambda^*)^2 Var\left(z_t \hat{\varepsilon}_t\right) + Var(\eta - \eta^*)
= \left(B_1 + \frac{\lambda^2 - \lambda^*}{2}\right)^2 \frac{\theta_z \sigma_z^2}{1 - \phi_z^2} + C_1^2 \frac{\sigma_x^2}{1 - \phi_x^2} + (\lambda - \lambda^*)^2 \theta_x + Var(\eta - \eta^*).
$$

Together these expressions give the autocorrelation of the exchange rate depreciation:

$$
Corr(\Delta s_{t+1}, \Delta s_t) = \frac{\left(B_1 + \frac{\lambda^2 - \lambda^*}{2}\right)^2 \phi_z \frac{\theta_z \sigma_z^2}{1 - \phi_z^2} + C_1^2 \phi_x \frac{\sigma_x^2}{1 - \phi_x^2} + \left(B_1 + \frac{\lambda^2 - \lambda^*}{2}\right)(\lambda - \lambda^*) \theta_z \sigma_z}{\left(B_1 + \frac{\lambda^2 - \lambda^*}{2}\right)^2 \phi_z \frac{\theta_z \sigma_z^2}{1 - \phi_z^2} + C_1^2 \phi_x \frac{\sigma_x^2}{1 - \phi_x^2} + (\lambda - \lambda^*)^2 \theta_x + Var(\eta - \eta^*)}.
$$

This expression does not seem to indicate any immediate restrictions on the autocorrelation of the exchange rate depreciation except when $(\lambda - \lambda^*) = 0$. Then the autocorrelation is restricted to nonnegative values.
A.3 Estimation and Identification

Because affine models rely on the assumption that disturbances are lognormal, I can use this assumption when estimating the models. Rather than estimating this model with the Generalized Method of Moments, as frequently done in the literature, I use the Maximum Likelihood approach to obtain the estimates. I use data on spot, one-month and three-month forward rates to construct exchange rate depreciation and forward premium.

I estimate both models using demeaned data. This simplification allows me to do away with several parameters that affect only the means of the variables and aren’t essential for the issues I discuss here. Below are estimation equations expressed with demeaned variables showing which parameters are lost after demeaning. I only show the equations for the two-state version (single-state equations can be easily deduced from the two-state model).

\[ f_t^1 - s_t - E( f_t^1 - s_t) = (B_1 - B_1^*) \hat{z}_t + C_1 \hat{x}_t, \]  
\[ f_t^3 - s_t - E( f_t^3 - s_t) = (B_3 - B_3^*) \hat{z}_t + C_3 \hat{x}_t, \]  
\[ \Delta s_{t+1} - E(\Delta s_{t+1}) = \left( (B_1 - B_1^*) + \frac{\lambda^2 - \lambda^{*2}}{2} \right) \hat{z}_t + C_1 \hat{x} + (\lambda - \lambda^*) (\theta z + \hat{z}_t)^{1/2} \varepsilon_{t+1} + \eta_{t+1} - \eta^*_{t+1}, \]  
\[ \hat{z}_{t+1} = \phi_z \hat{z}_t + \sigma_z (\theta z + \hat{z}_t)^{1/2} \varepsilon_{t+1}, \]  
\[ \hat{x}_{t+1} = \phi_x \hat{x}_t + \sigma_x \zeta_{t+1}. \]  

Parameters \((\gamma - \gamma^*) , \lambda, \lambda^*, \sigma_z \) and \(\sigma_x \) with \(\mu \) are identified only up to a factor of proportionality. I normalize \(\sigma_x \) and \(\sigma_z \) at 0.1. To see why these parameters are identified only up to a factor of proportionality, consider the state equation from the single-state model

\[ z_{t+1} = (1 - \phi) \theta + \phi z_t + \sigma \sqrt{z_t} \varepsilon_{t+1}. \]
Define $x = \frac{\tilde{x}_{t+1}}{\sigma^2}$ and substitute it into the state equation:

$$\sigma^2 x_{t+1} = (1 - \phi) \theta + \phi \sigma^2 x_t + \sigma \sqrt{\sigma^2 x_t} \epsilon_{t+1}.$$ 

Divide by $\sigma^2$

$$x_{t+1} = (1 - \phi) \theta' + \phi x_t + \sqrt{x_t} \epsilon_{t+1}$$

where $\theta' = \frac{\theta}{\sigma^2}$

Turning to the kernel equation:

$$-m_{t+1} = \delta + \left( \gamma + \frac{\lambda^2}{2} \right) \sigma^2 x_t + \lambda \left( \sigma^2 x_t \right)^{\frac{1}{2}} \epsilon_{t+1},$$

$$-m_{t+1} = \delta + \left( \gamma \sigma^2 + \left( \frac{\lambda \sigma}{2} \right)^2 \right) x_t + \lambda \sigma \left( x_t \right)^{\frac{1}{2}} \epsilon_{t+1},$$

$$-m_{t+1} = \delta + \left( \gamma' + \frac{\lambda'^2}{2} \right) x_t + \lambda' \left( x_t \right)^{\frac{1}{2}} \epsilon_{t+1}.$$ 

Therefore, after rescaling the state equation, the kernel equation retains the same form.
A.4 Estimation results and descriptive statistics

Table 1: Sample Moments and Fama Regression Estimates

<table>
<thead>
<tr>
<th></th>
<th>GBP/USD</th>
<th>DEM/USD</th>
<th>JPY/USD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E(s_t - s_{t-1})$</td>
<td>-0.0015</td>
<td>0.0001</td>
<td>0.0025</td>
</tr>
<tr>
<td>$E(f_t - s_t)$</td>
<td>-0.0019</td>
<td>0.0015</td>
<td>0.0026</td>
</tr>
<tr>
<td>$Var(s_t - s_{t-1})$</td>
<td>0.0010</td>
<td>0.0011</td>
<td>0.0012</td>
</tr>
<tr>
<td>$Var(f_t - s_t)$</td>
<td>$6.8 \times 10^{-6}$</td>
<td>$7.3 \times 10^{-6}$</td>
<td>$9.4 \times 10^{-6}$</td>
</tr>
<tr>
<td>$Corr(\Delta s_{t+1}, \Delta s_t)$</td>
<td>0.0664</td>
<td>0.0095</td>
<td>0.0509</td>
</tr>
<tr>
<td>$Corr(f_{t+1} - s_{t+1}, f_t - s_t)$</td>
<td>0.8067</td>
<td>0.8443</td>
<td>0.6682</td>
</tr>
</tbody>
</table>

The lower part of the table contains the estimates of the regression equation

$$\Delta s_{t+1} = \alpha_1 + \alpha_2 (f_t - s_t) + v_t.$$  

The numbers in parentheses are standard errors associated with each coefficient.
Table 2: Estimates of the single-factor model

<table>
<thead>
<tr>
<th></th>
<th>GBP/USD</th>
<th>DEM/USD</th>
<th>JPY/USD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi$</td>
<td>0.8259** (0.0204)</td>
<td>0.8943** (0.0154)</td>
<td>0.7002** (0.0209)</td>
</tr>
<tr>
<td>$\gamma - \gamma^*$</td>
<td>0.0114** (0.0015)</td>
<td>0.0130** (0.0010)</td>
<td>0.0168** (0.0024)</td>
</tr>
<tr>
<td>$\theta$</td>
<td>1.7910** (0.4675)</td>
<td>1.1539** (0.1559)</td>
<td>1.7686** (0.4614)</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>7.7091** (3.5744)</td>
<td>10.1799** (4.5274)</td>
<td>19.3618** (0.4554)</td>
</tr>
<tr>
<td>$\lambda^*$</td>
<td>7.7120** (3.5744)</td>
<td>10.1821** (4.5274)</td>
<td>19.3640** (0.4534)</td>
</tr>
<tr>
<td>$\text{Var}(\eta - \eta^*)$</td>
<td>0.0010** (0.0001)</td>
<td>0.0011** (0.0001)</td>
<td>0.0012** (0.0001)</td>
</tr>
<tr>
<td>$\frac{\lambda^2 - \lambda^{**}}{2}$</td>
<td>-0.0219** (0.0057)</td>
<td>-0.0218** (0.0030)</td>
<td>-0.0425** (0.0110)</td>
</tr>
<tr>
<td>$E(\Delta s_{t+1})$-implied</td>
<td>-0.0016 (0.0019)</td>
<td>0.0001 (0.0019)</td>
<td>0.0026 (0.0021)</td>
</tr>
<tr>
<td>$E(\Delta s_{t+1})$-sample</td>
<td>-0.0015</td>
<td>0.0001</td>
<td>0.0025</td>
</tr>
<tr>
<td>$\text{Corr}(\Delta s_t, \Delta s_{t-1})$-implied</td>
<td>0.0103 (0.0128)</td>
<td>0.0056 (0.0101)</td>
<td>0.0211 (0.0158)</td>
</tr>
<tr>
<td>$\text{Corr}(\Delta s_t, \Delta s_{t-1})$-sample</td>
<td>0.0664</td>
<td>0.0095</td>
<td>0.0509</td>
</tr>
<tr>
<td>$\text{Corr}(f_{t+1} - s_{t+1}, f_t - s_t)$-sample</td>
<td>0.8067</td>
<td>0.8443</td>
<td>0.6682</td>
</tr>
<tr>
<td>Fama coef. implied</td>
<td>-0.9272** (0.2529)</td>
<td>-0.6718** (0.2232)</td>
<td>-1.5252** (0.3573)</td>
</tr>
<tr>
<td>Fama coef. (OLS)</td>
<td>-0.8750 (0.6790)</td>
<td>-0.5896 (0.6763)</td>
<td>-1.4889** (0.6369)</td>
</tr>
</tbody>
</table>

Estimates obtained using MLE. The numbers in parentheses are standard errors associated with each coefficient. Standard errors are computed using the outer product of gradients of the likelihood function.
Table 3: Testing the implications of the single-factor model

1. Equation: $\psi_{t+1}^2 - \widetilde{Var}_t(\Delta s_{t+1}) = \gamma_0 + \gamma_1 (f_t + s_t) + \nu_t$

<table>
<thead>
<tr>
<th></th>
<th>GBP/USD</th>
<th>DEM/USD</th>
<th>JPY/USD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_0$</td>
<td>-0.0001 (0.0001)</td>
<td>0.0000 (0.0001)</td>
<td>-0.0003 (0.0002)</td>
</tr>
<tr>
<td>$\gamma_1$</td>
<td>-0.0491 (0.0452)</td>
<td>0.0208 (0.0367)</td>
<td>0.0926** (0.0409)</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.004</td>
<td>0.001</td>
<td>0.016</td>
</tr>
</tbody>
</table>

2. Equation: $\xi_{t+1}^2 - \widetilde{Var}_t(f_{t+1} - s_{t+1}) = \gamma_0 + \gamma_1 (f_t + s_t) + \nu_t$.

<table>
<thead>
<tr>
<th></th>
<th>GBP/USD</th>
<th>DEM/USD</th>
<th>JPY/USD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_0$</td>
<td>0.0000 (0.0000)</td>
<td>0.0000 (0.0000)</td>
<td>0.0000 (0.0000)</td>
</tr>
<tr>
<td>$\gamma_1$</td>
<td>0.0005** (0.0002)</td>
<td>0.0008** (0.0003)</td>
<td>0.0013** (0.0005)</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.019</td>
<td>0.028</td>
<td>0.019</td>
</tr>
</tbody>
</table>

3. Equation: $\psi_{t+1}\xi_{t+1} - \widetilde{Cov}_t(f_{t+1} - s_{t+1}, \Delta s_{t+1}) = \gamma_0 + \gamma_1 (f_t + s_t) + \nu_t$.

<table>
<thead>
<tr>
<th></th>
<th>GBP/USD</th>
<th>DEM/USD</th>
<th>JPY/USD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_0$</td>
<td>-0.0012** (0.0004)</td>
<td>-0.0001 (0.0002)</td>
<td>0.0002 (0.0003)</td>
</tr>
<tr>
<td>$\gamma_1$</td>
<td>-0.6059** (0.1135)</td>
<td>0.0601 (0.0654)</td>
<td>-0.0865 (0.0711)</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.082</td>
<td>0.003</td>
<td>0.005</td>
</tr>
</tbody>
</table>

The numbers in parentheses are standard errors associated with each coefficient.

Using parameter estimates, I construct innovations $\psi_{t+1}$ and $\xi_{t+1}$ of the exchange rate depreciation and the forward premium respectively. Squaring these innovations gives realizations of the variance, and computing their product gives the covariance of $\psi_{t+1}$ and $\xi_{t+1}$.

Then, I regress the difference between the realizations of the second moments and their conditional expectations on the forward premium which is just a linear function of the state variable $z_t$. 

The numbers in parentheses are standard errors associated with each coefficient.
Table 4: Moments of normalized innovations from the single-state model

<table>
<thead>
<tr>
<th></th>
<th>GBP/USD</th>
<th>DEM/USD</th>
<th>JPY/USD</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\varepsilon_t$</td>
<td>$\eta_t - \eta_t^*$</td>
<td>$\varepsilon_t$</td>
</tr>
<tr>
<td>Mean</td>
<td>0.0003</td>
<td>0.0000</td>
<td>0.0016</td>
</tr>
<tr>
<td>Variance</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>Skewness</td>
<td>0.5399</td>
<td>-0.0828</td>
<td>1.8414</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>18.8429</td>
<td>5.4150</td>
<td>36.0564</td>
</tr>
</tbody>
</table>
Empirical Test of Affine Stochastic Discount Factor Models of Currency Pricing

Table 5: Estimates of the two-factor model

<table>
<thead>
<tr>
<th></th>
<th>GBP/USD</th>
<th>DEM/USD</th>
<th>JPY/USD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_z$</td>
<td>0.1676** (0.0087)</td>
<td>0.0037** (0.0096)</td>
<td>0.0120 (0.0249)</td>
</tr>
<tr>
<td>$\phi_x$</td>
<td>0.9581** (0.0034)</td>
<td>0.9661** (0.0067)</td>
<td>0.9312** (0.0114)</td>
</tr>
<tr>
<td>$\gamma - \gamma^*$</td>
<td>0.0306** (0.0010)</td>
<td>0.0207** (0.0008)</td>
<td>-0.0103** (0.0003)</td>
</tr>
<tr>
<td>$\theta$</td>
<td>0.0958** (0.0032)</td>
<td>0.2018** (0.0079)</td>
<td>2.2467** (0.0507)</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>-16.7114** (0.0077)</td>
<td>26.2357** (0.0680)</td>
<td>16.7553** (0.0111)</td>
</tr>
<tr>
<td>$\lambda^*$</td>
<td>-16.7056** (0.0036)</td>
<td>26.2377** (0.0176)</td>
<td>16.7546** (0.0145)</td>
</tr>
<tr>
<td>$\mu$</td>
<td>-0.0099** (0.0001)</td>
<td>-0.0076** (0.0001)</td>
<td>-0.0114** (0.0001)</td>
</tr>
<tr>
<td>$\text{Var}(\eta - \eta^*)$</td>
<td>0.0010 (0.0021)</td>
<td>0.0011 (0.0022)</td>
<td>0.0013 (0.0022)</td>
</tr>
</tbody>
</table>

Estimates obtained using MLE. The numbers in parentheses are standard errors associated with each coefficient. Standard errors are computed using the outer product of gradients of the likelihood function.
Table 6: Testing the implications of the two-factor model

1. Equation: $\psi_{t+1}^2 - \text{Var}_t(\Delta s_{t+1}) = \gamma_0 + \gamma_1 z_t + \nu_t$.

<table>
<thead>
<tr>
<th></th>
<th>GBP/USD</th>
<th>DEM/USD</th>
<th>JPY/USD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_0$</td>
<td>-0.0000 (0.0001)</td>
<td>-0.0000 (0.0001)</td>
<td>-0.0000 (0.0001)</td>
</tr>
<tr>
<td>$\gamma_1$</td>
<td>-0.1393 (3.8021)</td>
<td>0.8786 (2.1334)</td>
<td>-0.1535 (0.8238)</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.000</td>
<td>0.001</td>
<td>0.000</td>
</tr>
</tbody>
</table>

2. Equation: $\xi_{t+1}^2 - \text{Var}_t(f_{t+1} - s_{t+1}) = \gamma_0 + \gamma_1 z_t + \nu_t$.

<table>
<thead>
<tr>
<th></th>
<th>GBP/USD</th>
<th>DEM/USD</th>
<th>JPY/USD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_0$</td>
<td>0.0000 (0.0000)</td>
<td>0.0000 (0.0000)</td>
<td>0.0000 (0.0000)</td>
</tr>
<tr>
<td>$\gamma_1$</td>
<td>0.0059 (0.0160)</td>
<td>-0.0011 (0.0136)</td>
<td>-0.0011 (0.0097)</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
</tbody>
</table>

3. Equation: $\psi_{t+1} \xi_{t+1} - \text{Cov}_t(f_{t+1} - s_{t+1}, \Delta s_{t+1}) = \gamma_0 + \gamma_1 z_t + \nu_t$.

<table>
<thead>
<tr>
<th></th>
<th>GBP/USD</th>
<th>DEM/USD</th>
<th>JPY/USD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_0$</td>
<td>0.0000 (0.0001)</td>
<td>0.0000 (0.0001)</td>
<td>0.0000 (0.0002)</td>
</tr>
<tr>
<td>$\gamma_1$</td>
<td>-2.2639 (1.6278)</td>
<td>0.7138 (1.4200)</td>
<td>0.7988 (1.3972)</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.006</td>
<td>0.001</td>
<td>0.001</td>
</tr>
</tbody>
</table>

The numbers in parentheses are standard errors associated with each coefficient.

Using parameter estimates, I construct innovations $\psi_{t+1}$ and $\xi_{t+1}$ of the exchange rate depreciation and the forward premium respectively. Squaring these innovations gives realizations of the variance, and computing their product gives the covariance of $\psi_{t+1}$ and $\xi_{t+1}$. Then, I regress the difference between the realizations of the second moments and their conditional expectations on the state variable $z_t$. 

<table>
<thead>
<tr>
<th></th>
<th>GBP/USD</th>
<th>DEM/USD</th>
<th>JPY/USD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.0090</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>Variance</td>
<td>1.0188</td>
<td>1.0000</td>
<td>0.0321</td>
</tr>
<tr>
<td>Skewness</td>
<td>5.2081</td>
<td>0.6533</td>
<td>0.0001</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>55.7847</td>
<td>8.2684</td>
<td>0.0054</td>
</tr>
</tbody>
</table>

Table 7: Moments of innovations from the two state model